

## ON THE INVERTIBILITY OF QUANTIZATION FUNCTORS

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ABSTRACT. Certain quantization problems are equivalent to the construction of morphisms from "quantum" to "classical" props. Once such a morphism is constructed, Hensel's lemma shows that it is in fact an isomorphism. This gives a new, simple proof that any Etingof-Kazhdan quantization functor is an equivalence of categories between quantized universal enveloping (QUE) algebras and Lie bialgebras over a formal series ring (dequantization). We apply the same argument to construct dequantizations of formal solutions of the quantum Yang-Baxter equation and of quasitriangular QUE algebras. We also give structure results for the props involved in quantization of Lie bialgebras, which yield an associator-independent proof that the prop of QUE algebras is a flat deformation of the prop of co-Poisson universal enveloping algebras.

## 1. INTRODUCTION

A prop ("product and permutation category") is an algebraic object generalizing the notion of an operad (see [M]). Given a symmetric monoidal category  $\mathcal{S}$ , and a prop  $\underline{P}$ , one can define the category of  $\underline{P}$ -modules over  $\mathcal{S}$ ,  $\text{Mod}_{\mathcal{S}}(\underline{P})$ . A morphism of props  $\underline{P} \rightarrow \underline{Q}$  then gives rise to a functor  $\text{Mod}_{\mathcal{S}}(\underline{Q}) \rightarrow \text{Mod}_{\mathcal{S}}(\underline{P})$ .

In quantization problems, one should define functors from "classical" to "quantum" categories, left inverse to the "semiclassical limit" functor. Explicitly, let  $\mathcal{C}_{\text{class}}$  and  $\mathcal{C}_{\text{quant}}$  be these categories, and  $\text{SC} : \mathcal{C}_{\text{quant}} \rightarrow \mathcal{C}_{\text{class}}$  be the semiclassical limit functor. Then  $Q : \mathcal{C}_{\text{class}} \rightarrow \mathcal{C}_{\text{quant}}$  is a quantization functor if  $\text{SC} \circ Q = \text{id}$ .

In some cases, we have props  $\underline{P}_{\text{class}}$  and  $\underline{P}_{\text{quant}}$ , such that  $\mathcal{C}_x = \text{Mod}_{\mathcal{S}}(\underline{P}_x)$  for  $x = \text{class}$  or  $\text{quant}$ . We denote the base field by  $\mathbb{K}$ , and by  $\hbar$  a formal variable; then  $\underline{P}_{\text{quant}}$  is a module over  $\mathbb{K}[[\hbar]]$ , whereas the base ring for  $\underline{P}_{\text{class}}$  is  $\mathbb{K}$ . Modules over the prop  $\underline{P}_{\text{quant}}/(\hbar)$  are provided by  $V/(\hbar)$ , where  $V$  is an object of  $\mathcal{C}_{\text{quant}}$ . Such an object carries a classical structure, and is therefore a  $\underline{P}_{\text{class}}$ -module. This operation has a propic interpretation: we have a prop morphism  $\underline{\text{SC}} : \underline{P}_{\text{class}} \rightarrow \underline{P}_{\text{quant}}/(\hbar)$  inducing  $\text{SC}$ . Modules over  $\underline{P}_{\text{class}}[[\hbar]]$  are provided by  $\hbar$ -dependent analogues of the objects of  $\mathcal{C}_{\text{class}}$ ; e.g., by the  $V[[\hbar]]$ , where  $V \in \text{Ob}(\mathcal{C}_{\text{class}})$  (here the structure maps are  $\hbar$ -independent).

Then a quantization functor  $\mathcal{C}_{\text{class}} \rightarrow \mathcal{C}_{\text{quant}}$  may be obtained from a prop morphism  $Q : \underline{P}_{\text{quant}} \rightarrow \underline{P}_{\text{class}}[[\hbar]]$ , such that  $(Q \bmod \hbar) \circ \underline{\text{SC}}$  is the identity of  $\underline{P}_{\text{class}}$ . We call such a  $Q$  a *quantization morphism*. (Some quantization problems, like quantization of Poisson manifolds or algebras, do not fit into this scheme, see Remark 2.)

The main observation of this paper is the following. Assume in addition that  $\underline{\text{SC}}$  is surjective. Then Hensel's lemma implies that  $\underline{\text{SC}}$  and  $Q$  are isomorphisms. Therefore the set of quantization morphisms is a torsor, with underlying groups  $\text{Aut}_1(\underline{P}_{\text{quant}})$  and  $\text{Aut}_1(\underline{P}_{\text{class}})$ , the subgroups of automorphisms of  $\underline{P}_{\text{quant}}$  and  $\underline{P}_{\text{class}}[[\hbar]]$  whose reduction modulo  $\hbar$  is the identity. Moreover, any quantization morphism yields an equivalence of categories between  $\mathcal{C}_{\text{quant}}$  and  $\mathcal{C}_{\text{class}, \hbar} = \text{Mod}_{\mathcal{S}}(\underline{P}_{\text{class}}[[\hbar]])$ , i.e., between the quantum category and the  $\hbar$ -dependent version of the classical category. We call this a dequantization result.

We apply this to the following three situations: (1) quantization of solutions of the CYBE (classical Yang-Baxter equation), (2) quantization of Lie bialgebras, (3) quantization of quasitriangular Lie bialgebras. Dequantization in situation (2) was first obtained in [EK2] using the group GT.

All three cases are direct applications of the above argument, combined in the two last cases with the co-Poisson, or quasitriangular versions of the Milnor-Moore theorem.

In the second situation, we also give an explicit description of the structure of the props involved. (A simple description of the props involved seems to be impossible in the two other cases.) In particular, we prove directly (i.e., not using the existence of quantization functors) that the prop QUE of QUE algebras is a flat deformation of the prop UE<sub>cP</sub> of co-Poisson universal enveloping algebras. This implies that any morphism QUE  $\rightarrow$  UE<sub>cP</sub>[[ $\hbar$ ]], whose reduction modulo  $\hbar$  is the identity, is an isomorphism. (This by itself does not imply the existence of quantization functors, see Remark 2.)

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## 2. THE FORMALISM OF PROPS

**2.1. Definition, properties.** We fix a base field  $\mathbb{K}$  of characteristic zero, and a base ring  $R$  containing  $\mathbb{K}$ ; which will be either  $\mathbb{K}[[\hbar]]$  or  $\mathbb{K}$  itself. The modules over  $\mathbb{K}[[\hbar]]$  will always be quotients of topologically free modules by closed submodules, and their direct sums and tensor products will be understood in this category; the maps between them will always be continuous.

A prop over  $R$  is a symmetric monoidal category  $\mathcal{C}$  generated by one object  $O$ . All the information about such a category is contained in the  $R$ -modules  $\text{Hom}_{\mathcal{C}}(O^{\otimes p}, O^{\otimes q})$ ,  $p, q \geq 0$  and the operations relating them. More specifically, we have:

**Definition 2.1.** (see [M, L]) A prop  $\underline{P}$  over  $R$  is a collection of  $R$ -modules  $\underline{P}(n, m)$ ,  $n, m \geq 0$ , together with the data of:

- (1)  $R$ -module maps  $\circ : \underline{P}(n, m) \otimes \underline{P}(m, p) \rightarrow \underline{P}(n, p)$  and  $\otimes : \underline{P}(n, m) \otimes \underline{P}(n', m') \rightarrow \underline{P}(n + n', m + m')$ , denoted  $f \otimes g \mapsto g \circ f$  and  $f \otimes g \mapsto f \otimes g$ ,
- (2) linear maps  $i_n : \mathbb{Q}\mathfrak{S}_n \rightarrow \underline{P}(n, n)$ ,  $n \geq 0$ , such that
  - (a)  $\circ$  and  $\otimes$  are associative:  $(x \circ y) \circ z = x \circ (y \circ z)$  and  $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ . Moreover, we have  $(x \circ x') \otimes (y \circ y') = (x \otimes y) \circ (x' \otimes y')$ ,
  - (b)  $i_n$  is an algebra morphism from  $\mathbb{Q}\mathfrak{S}_n$  to  $(\underline{P}(n, n), \circ)$ ,
  - (c) for  $\sigma \in \mathfrak{S}_n$  and  $\sigma' \in \mathfrak{S}_{n'}$ , denote by  $\sigma * \sigma'$  the permutation of  $\mathfrak{S}_{n+n'}$  such that  $(\sigma * \sigma')(i) = \sigma(i)$  for  $i \leq n$  and  $(\sigma * \sigma')(i) = \sigma'(i - n) + n$  for  $i > n$ . Then  $i_{n+n'}(\sigma * \sigma') = i_n(\sigma) \otimes i_{n'}(\sigma')$
  - (d) if we set  $\text{id} = i_1(e)$  ( $e$  is the only element of  $\mathfrak{S}_1$ ), then the identity  $\text{id}^{\otimes m} \circ x = x \circ \text{id}^{\otimes n} = x$  holds for  $x \in \underline{P}(n, m)$
  - (e) if  $\sigma_{n,n'}$  is the permutation in  $\mathfrak{S}_{n+n'}$  such that  $\sigma_{n,n'}(i) = i + n'$  for  $i = 1, \dots, n$  and  $\sigma_{n,n'}(i) = i - n$  for  $i = n + 1, \dots, n + n'$ , and if  $x \in \underline{P}(n, m)$  and  $y \in \underline{P}(n', m')$ , then

$$y \otimes x = \sigma_{m,m'} \circ (x \otimes y) \circ \sigma_{n',n}.$$

If  $\mathcal{C}$  is a symmetric monoidal category generated by  $O$ , then the corresponding prop  $\underline{P}_{\mathcal{C}}$  is such that  $\underline{P}_{\mathcal{C}}(n, m) = \text{Hom}_{\mathcal{C}}(O^{\otimes n}, O^{\otimes m})$ .

If  $\underline{P}$  and  $\underline{Q}$  are two props, then a morphism  $\phi : \underline{P} \rightarrow \underline{Q}$  is a collection of  $R$ -module maps  $\phi(n, m) : \underline{P}(n, m) \rightarrow \underline{Q}(n, m)$ , such that the natural diagrams commute.

An ideal  $\underline{I}$  of  $\underline{P}$  is a collection of  $R$ -submodules  $\underline{I}(n, m) \subset \underline{P}(n, m)$ , such that  $\underline{I}(n, m) \circ \underline{P}(m, p) \subset \underline{I}(n, p)$ ,  $\underline{P}(n, m) \circ \underline{I}(m, p) \subset \underline{I}(n, p)$ , and  $\otimes$  takes both  $\underline{I}(n, m) \otimes \underline{P}(n', m')$  and  $\underline{P}(n, m) \otimes \underline{I}(n', m')$  to  $\underline{I}(n + n', m + m')$ . The collection of kernels defined by a prop morphism

is a prop ideal. An ideal  $\underline{I}$  of a prop  $\underline{P}$  gives rise to a quotient prop  $\underline{P}/\underline{I}$ , defined by  $(\underline{P}/\underline{I})(p, q) = \underline{P}(p, q)/\underline{I}(p, q)$ .

If  $\underline{P}$  be a prop over  $R$ , then the collection of all torsion submodules  $\underline{P}(p, q)_{\text{tor}} \subset \underline{P}(p, q)$  is an ideal of  $\underline{P}$ . We call it the torsion ideal.

$\underline{P}$  is a topological prop if it is equipped with a decreasing family  $\underline{I}_n$  of prop ideals. We then say that the sequence  $x_n \in \oplus_{p,q} \underline{P}(p, q)$  tends to zero if  $x_n \in \oplus_{p,q} \underline{I}_{k(n)}(p, q)$ , where  $k(n)$  goes to infinity with  $n$ .

We will use the following notation. If  $x_1, \dots, x_p$  are such that  $x_i \in \underline{P}(0, n_i)$ , if  $n = \sum_i n_i$  and  $(I_1, \dots, I_p)$  is a partition of  $[1, n]$  by ordered sets  $I_1, \dots, I_p$ , then  $x_1^{I_1} \cdots x_p^{I_p}$  is the element of  $\underline{P}(0, n)$  equal to  $\sigma \circ (x_1 \otimes \cdots \otimes x_p)$ , where  $\sigma \in \mathfrak{S}_n$  is the block permutation attached to  $I_1, \dots, I_p$ . E.g.,  $x^{1,4} y^{3,2} = (1432) \circ (x \otimes y)$ . (We denote by  $(i_1 \dots i_k)$  the permutation taking 1 to  $i_1$ , ...,  $k$  to  $i_k$ .)

**2.2. Props and operads.** Any operad gives rise to a prop. If  $(O(n))_{n \geq 0}$  is the family of  $\mathfrak{S}_n$ -modules underlying an operad, then the vector spaces underlying the corresponding prop are

$$O(n, m) = \bigoplus_{(I_1, \dots, I_m) \in \text{Part}_m(n)} O(\text{card}(I_1)) \otimes \cdots \otimes O(\text{card}(I_m)). \quad (1)$$

Here  $\text{Part}_m(n)$  is the set of partitions of  $[1, n]$  by  $m$  unordered sets. So  $O(n, m)$  vanishes unless  $n \geq m$ . A similar construction holds with cooperads.

### 2.3. Props defined by generators and relations.

**Lemma 2.1.** *If  $V = V(n, m), n, m \geq 0$  is a collection of vector spaces, then there is a pair  $(\underline{P}_V, \alpha_V)$  of a prop  $\underline{P}_V$  and a collection of linear maps  $\alpha_{V,n,m} : V(n, m) \rightarrow \underline{P}_V(n, m)$ , with the following universal property. If  $(\underline{P}, \alpha)$  is any pair of a prop  $\underline{P}$  and a collection of linear maps  $\alpha_{n,m} : V(n, m) \rightarrow \underline{P}(n, m)$ , then there is a unique prop morphism  $\alpha_P : \underline{P}_V \rightarrow \underline{P}$ , such that  $\alpha_P \circ \alpha_V = \alpha$ .  $\underline{P}_V$  is unique up to isomorphism, we call it the free prop generated by  $V$ .*

*Proof.* We construct  $\underline{P}_V$  as follows. Choose a basis  $(e_{i,j}^\alpha)_\alpha$  of each  $V(i, j)$ . For each  $n, m$ , let  $G_V(n, m)$  be the set of oriented graphs  $\Gamma$  of the following type. Vertices of  $\Gamma$  are of three types: "inputs", "outputs" and "operations". "Operations" vertices correspond to an index  $(i, j, \alpha)$ . A vertex is said to be of valency  $(p, q)$  if it has  $p$  incoming and  $q$  outgoing edges. Input, output and  $(i, j, \alpha)$  vertices are of valency  $(0, 1)$ ,  $(1, 0)$  and  $(i, j)$ . Each vertex carries an order of its input and output edges.  $\Gamma$  has no oriented cycle. Then  $\underline{P}_V(n, m)$  is the topologically free module spanned by  $G_V(n, m)$ . We define a map  $\mathfrak{S}_n \rightarrow G_V(n, n)$ , taking  $\sigma$  to the graph of  $n$  empty edges with (incoming label, outgoing label)  $= (i, \sigma(i))$ . It extends to a linear map  $\mathbb{K}\mathfrak{S}_n \rightarrow \underline{P}_V(n, n)$ . There are unique maps

$$\circ^{\text{graphs}} : G_V(n, m) \times G_V(m, p) \rightarrow G_V(n, p)$$

and

$$\otimes^{\text{graphs}} : G_V(n, m) \times G_V(n', m') \rightarrow G_V(n + n', m + m')$$

defined as follows. If  $\Gamma$  and  $\Gamma'$  are graphs, then  $\circ^{\text{graphs}}(\Gamma, \Gamma')$  is obtained from  $\Gamma$  and  $\Gamma'$  by connecting the output vertex of  $\Gamma$  with the input vertex of  $\Gamma'$  with the same index, and then deleting the input and output vertices, and  $\otimes^{\text{graphs}}(\Gamma, \Gamma')$  is obtained from  $\Gamma$  and  $\Gamma'$  by adding  $n$  (resp.,  $m$ ) to the index of each input (resp., output) vertex of  $\Gamma'$ . Then  $\circ$  and  $\otimes$  are the linear maps extending  $\circ^{\text{graphs}}$  and  $\otimes^{\text{graphs}}$ .  $\square$

Let  $V$  be given, and let  $\mathcal{R}$  be a graded  $R$ -submodule of  $\oplus_{n,m} \underline{P}_V(n, m)$ . We set  $\mathcal{R}(n, m) = \mathcal{R} \cap \underline{P}_V(n, m)$ , so  $\mathcal{R} = \oplus_{n,m} \mathcal{R}(n, m)$ . Then we have

**Lemma 2.2.** *There exists a unique pair  $(\underline{P}_{V,\mathcal{R}}, \text{can})$  of a prop  $\underline{P}_{V,\mathcal{R}}$  and a prop morphism  $\text{can} : \underline{P}_V \rightarrow \underline{P}_{V,\mathcal{R}}$ , such that  $\text{can}(\mathcal{R}) = 0$ , with the following property. If  $(\underline{Q}, \beta)$  is a pair of a prop  $\underline{Q}$  and a prop morphism  $\beta : \underline{P}_V \rightarrow \underline{Q}$ , such that  $\beta(\mathcal{R}) = 0$ , then there is a unique prop morphism  $\gamma : \underline{P}_{V,\mathcal{R}} \rightarrow \underline{Q}$ , such that  $\gamma \circ \text{can} = \beta$ .*

*Proof.* There is a smallest ideal  $\underline{L}_{\mathcal{R}}$  of  $\underline{P}_V$  (the ideal generated by  $\mathcal{R}$ ), such that  $\mathcal{R} \subset \oplus_{p,q} \underline{L}_{\mathcal{R}}(p, q)$ . We then set  $\underline{P}_{V,\mathcal{R}}(n, m) = \underline{P}_V(n, m) / \underline{L}_{\mathcal{R}}(n, m)$ .  $\square$

Let us say that two linear combinations of graphs of  $G_V(n, m)$  are equivalent if their difference is a linear combination of substitutions of diagrams of  $\mathcal{R}$  in given graphs. Then this equivalence relation is compatible with the prop structure, and  $\underline{P}_{V,\mathcal{R}}(n, m)$  is the quotient of  $\underline{P}_V(n, m)$  by this equivalence relation.

If  $\underline{P}$  is a prop defined by generators and relations, and  $\mathcal{R}'$  is a collection of new relations involving  $x_1, x_2, \dots$  and the generators of  $\underline{P}$ , we define  $\underline{P}\langle x_1, x_2, \dots \rangle / (\mathcal{R}')$  as the prop with generators  $\{\text{generators of } \underline{P}\} \cup \{x_1, x_2, \dots\}$  and relations  $\{\text{relations of } \underline{P}\} \cup \mathcal{R}'$  (this definition is actually independent on the presentation of  $\underline{P}$ ).

*Remark 1.* Any algebra  $A$  gives rise to a prop  $P_A$ , where we define  $P_A(n, n)$  as the semidirect product of  $A^{\otimes n}$  with  $\mathfrak{S}_n$ , acting on  $A^{\otimes n}$  by permutation of factors, and  $P_A(n, m) = 0$  if  $n \neq m$ ;  $\circ$  is the product in  $A^{\otimes n} \rtimes \mathfrak{S}_n$  and  $\otimes$  is the product of the tensor product and the natural map  $\mathfrak{S}_n \times \mathfrak{S}_{n'} \rightarrow \mathfrak{S}_{n+n'}$ . The presentation of a prop by generators and relations is then a generalization of the similar notion in the case of algebras.

**2.4. Modules over props.** Let  $\mathcal{S}$  be a symmetric monoidal category over  $R$ . Then if  $A$  is any object in  $\mathcal{S}$ , the standard operations define a prop  $\underline{\text{Prop}}(A)$ , where  $\underline{\text{Prop}}(A)(p, q) = \text{Hom}(A^{\otimes p}, A^{\otimes q})$ . A structure of  $\underline{P}$ -module over a prop  $\underline{P}$  is a pair  $(A, \rho)$  of an object  $A$  of  $\mathcal{S}$  and a prop morphism  $\rho : \underline{P} \rightarrow \underline{\text{Prop}}(A)$ . A morphism between two  $\underline{P}$ -modules  $(A, \rho)$  and  $(B, \rho')$  is a morphism  $\lambda : A \rightarrow B$  in  $\mathcal{S}$ , such that if  $x \in \underline{P}(p, q)$  and  $a \in A^{\otimes p}$ ,  $\lambda^{\otimes q}(\rho(x)(a)) = \rho'(x)(\lambda^{\otimes p}(a))$ . Then  $\underline{P}$ -modules form a category.

We will sometimes denote  $\rho(x) \in \text{Hom}(A^{\otimes p}, A^{\otimes q})$  by  $x_A$ .

If  $\underline{P}$  is topological, we require that the map  $\oplus_{p,q} \underline{P}(p, q) \rightarrow \text{End}(\oplus_p A^{\otimes p})$  be continuous in the weak topology: if  $x_n \in \oplus_{p,q} \underline{P}(p, q)$  tends to zero, and  $a \in \hat{\bigoplus}_{n \geq 0} A^{\otimes n}$ , then  $\rho(x_n)(a)$  tends to zero as  $n \rightarrow \infty$ . Here  $\hat{\bigoplus}$  denotes the completed direct sum (direct product). When  $\mathcal{S}$  is the category of  $R$ -modules and  $R = \mathbb{K}$ , this means that  $\rho(x_n)(a)$  vanishes for  $n$  large enough.

**2.5. Operations on props.** Let  $\text{Irr}(n)$  be the set of conjugacy classes of primitive idempotents of  $\mathbb{Q}\mathfrak{S}_n$ . Let  $\pi \in \text{Irr}(n)$ , let  $\bar{\pi} \in \mathbb{Q}\mathfrak{S}_n$  be a representative of  $\pi$ . The corresponding simple Schur functor  $F_{(n,\pi)} : \text{Vect} \rightarrow \text{Vect}$  is defined by  $F_{(n,\pi)}(V) = \pi(V^{\otimes n})$ . A Schur functor is defined by a multiplicity map  $\mu : \coprod_{n \geq 0} \text{Irr}(n) \rightarrow \mathbb{Z}_{\geq 0}$ . Then  $F_{\mu}(V) := \oplus_{n \geq 0, \pi \in \text{Irr}(n)} F_{(n,\pi)}(V)^{\oplus \mu(n,\pi)}$ . The tensor product of two Schur functors  $F, G$  is defined by  $(F \otimes G)(V) = F(V) \otimes G(V)$ .

If  $\underline{P}$  is a prop and  $F = F_{\mu}$ ,  $F' = F_{\mu'}$  are Schur functors, we set

$$\underline{P}(F, F') = \widehat{\bigoplus_{n \geq 0, \pi \in \text{Irr}(n)} \bigoplus_{n' \geq 0, \pi' \in \text{Irr}(n')} (\pi' \circ \underline{P}(n, n') \circ \pi)^{\oplus \mu(n,\pi) \mu'(n', \pi')}}.$$

If  $F$  is a Schur functor, we define the prop  $F(\underline{P})$  by  $F(\underline{P})(p, q) = \underline{P}(F^{\otimes p}, F^{\otimes q})$ . Then the  $F(A)$ ,  $A \in \text{Mod}_{\mathcal{S}}(\underline{P})$ , are modules over  $F(\underline{P})$ .

### 3. DEQUANTIZATION OF SOLUTIONS OF QYBE

We denote by CYBA the prop defined over  $\mathbb{K}$  by generators  $\eta, m, r$  of bidegrees  $(0, 1)$ ,  $(2, 1)$ ,  $(0, 2)$ , and the following relations:  $m \circ (\eta \otimes \text{id}) = m \circ (\text{id} \otimes \eta) = \text{id}$ ,  $m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)$ ,

$$(\mu \otimes \text{id} \otimes \text{id})(r^{1,3} r^{2,4}) + (\text{id} \otimes \mu \otimes \text{id})(r^{1,2} r^{3,4}) + (\text{id} \otimes \text{id} \otimes \mu)(r^{1,3} r^{2,4}) = 0 \quad (2)$$

where  $\mu = m - m \circ (21)$  (the first relations mean that we have a prop morphism  $\underline{\text{Alg}} \rightarrow \underline{\text{CYBA}}$ , taking  $\eta, m$  to their analogues).

Let  $\mathcal{S}$  be the category of vector spaces, then  $\text{Mod}_{\mathcal{S}}(\underline{\text{CYBA}})$  is the category of quadruples  $(A, m_A, 1, r_A)$  of an associative algebra  $(A, m_A, 1)$  with unit, together with a solution  $r_A \in A^{\otimes 2}$  of the CYBE:  $\text{CYB}(r_A) := [r_A^{1,2}, r_A^{1,3}] + [r_A^{1,2}, r_A^{2,3}] + [r_A^{1,3}, r_A^{2,3}] = 0$ .

We denote by  $\underline{\text{QYBA}}$  the quotient of the free prop over  $\mathbb{K}[[\hbar]]$  generated by  $\eta, m, \rho$  of bidegrees  $(0, 1)$ ,  $(2, 1)$ ,  $(0, 2)$ , by the  $\hbar$ -adically closed ideal generated by (a) the same relations as above between  $\eta$  and  $m$ , (b) the relation

$$\begin{aligned} &(\mu \otimes \text{id} \otimes \text{id})(\rho^{1,3} \rho^{2,4}) + (\text{id} \otimes \mu \otimes \text{id})(\rho^{1,2} \rho^{3,4}) + (\text{id} \otimes \text{id} \otimes \text{id})(\rho^{1,3} \rho^{2,4}) \\ &+ \hbar(m \otimes m \otimes m)(\rho^{1,3} \rho^{2,5} \rho^{4,6} - \rho^{3,5} \rho^{1,6} \rho^{2,4}) = 0 \end{aligned}$$

where  $\mu = m - m \circ (21)$ .

Let  $\mathcal{S}_{\hbar}$  be the category of topologically free  $\mathbb{K}[[\hbar]]$ -modules, then  $\text{Mod}_{\mathcal{S}_{\hbar}}(\underline{\text{QYBA}})$  is the category of quadruples  $(B, m_B, 1, \rho_B)$ , where  $(B, m_B, 1)$  is a topologically free algebra, together with  $\rho_B \in B^{\otimes 2}$ , such that  $\text{CYB}(\rho_B) + \hbar(\rho_B^{1,2} \rho_B^{1,3} \rho_B^{2,3} - \rho_B^{2,3} \rho_B^{1,3} \rho_B^{1,2}) = 0$ . This equation is equivalent to the condition that  $R_B = 1 + \hbar \rho_B$  satisfies the QYBE (quantum Yang-Baxter equation).

Now we have a prop morphism  $\underline{\text{SC}} : \underline{\text{CYBA}} \rightarrow \underline{\text{QYBA}}/(\hbar)$ , taking  $\eta, m, r$  to the classes of  $\eta, m, \rho$ . The props  $\underline{\text{CYBA}}$  and  $\underline{\text{QYBA}}/(\hbar)$  have the same presentation, therefore  $\underline{\text{SC}}$  is an isomorphism. On the other hand, according to [EK, EK2], there exists a prop morphism  $\underline{Q} : \underline{\text{QYBA}} \rightarrow \underline{\text{CYBA}}[[\hbar]]$ , such that  $(\underline{Q} \bmod \hbar) \circ \underline{\text{SC}}$  is the identity. This means that  $(\underline{Q} \bmod \hbar)$  is  $\underline{\text{SC}}^{-1}$ .

Recall Hensel's lemma:

**Lemma 3.1.** *If  $N$  is a quotient of a topologically free  $\mathbb{K}[[\hbar]]$ -module by a closed submodule,  $M$  is a vector space, and  $f : N \rightarrow M[[\hbar]]$  is a continuous linear map such that  $(f \bmod \hbar)$  is an isomorphism, then  $f$  is an isomorphism. In particular,  $N$  is torsion-free.*

Applying this lemma to the collection of all  $\underline{\text{CYBA}}(p, q)$  and  $\underline{\text{QYBA}}(p, q)$ , we find:

**Proposition 3.1.**  *$\underline{Q} : \underline{\text{QYBA}} \rightarrow \underline{\text{CYBA}}[[\hbar]]$  is an isomorphism of props.*

Recall that  $\underline{Q}$  takes  $m$  to its analogue. For each topologically free algebra  $(A, m_A, 1)$  over  $\mathbb{K}[[\hbar]]$ , we have therefore a map  $r_A \mapsto R(r_A)$  from  $\{r_A \in A^{\otimes 2} \mid r_A \text{ satisfies the CYBE}\}$  to  $\{R_A \in 1 + \hbar A^{\otimes 2} \mid R_A \text{ satisfies the QYBE}\}$ , such that  $R(r_A) = 1 + \hbar r_A + O(\hbar^2)$ . Here  $\rho(r_A) = (R(r_A) - 1)/\hbar$  is given by a series  $r_A + \sum_{k \geq 2} \hbar^k P_k(m_A, r_A)$ , where each  $P_k$  a certain "polynomial" in  $m_A, r_A$ . For instance,  $P_k$  could be equal to  $\sum_{i,j} a_i a_j b_i \otimes b_j$ , where  $r_A = \sum_i a_i \otimes b_i$ . It is easy to show that such a series can be "triangularly" inverted, writing  $r_A = \rho - \sum_{k \geq 1} \hbar^k P_k(m_A, r_A)$  and substituting this expression in this identity iteratively.

Moreover, we know that the  $P_k$  can be chosen to be "normally ordered", i.e., in each tensor factor the components  $a_i$  are at the left of the components  $b_j$  (in the language of [Enr2],  $P_k$  belongs to  $(U(\mathfrak{g})^{\otimes 2})_{\text{univ}}$ ).

**Corollary 3.1.** *The assignment  $r_A \mapsto R(r_A)$  sets up a bijection between  $\{\text{solutions of CYBE in } A^{\otimes 2}\}$  and  $\{\text{solutions of QYBE in } 1 + \hbar A^{\otimes 2}\}$ .*

#### 4. DEQUANTIZATION OF QUE ALGEBRAS

**4.1. The prop  $\underline{\text{Bialg}}$  and related props.** We denote by  $\underline{\text{Bialg}}$  the prop of bialgebras. It is defined over  $\mathbb{K}$  by generators  $m, \Delta, \eta, \epsilon$  of bidegrees  $(2, 1)$ ,  $(1, 2)$ ,  $(0, 1)$ ,  $(1, 0)$  and relations

$$\begin{aligned} m \circ (m \otimes \text{id}) &= m \circ (\text{id} \otimes m), \quad m \circ (\text{id} \otimes \eta) = m \circ (\eta \otimes \text{id}) = \text{id}, \\ (\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta, \quad (\epsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \epsilon) \circ \Delta = \text{id}, \end{aligned}$$

$$\Delta \circ m = (m \otimes m) \circ (1324) \circ (\Delta \otimes \Delta), \quad \Delta \circ \eta = \eta \otimes \eta, \quad \epsilon \circ m = \epsilon \otimes \epsilon.$$

If  $\mathcal{S}$  is the category of  $\mathbb{K}$ -vector spaces, then  $\text{Mod}_{\mathcal{S}}(\underline{\text{Bialg}})$  is the category of bialgebras over  $\mathbb{K}$ .

We define the prop  $\underline{\text{Bialg}}_{\text{qcoco}}$  as the quotient of the prop  $\underline{\text{Bialg}}[[\hbar]]\langle\tilde{\delta}\rangle/(\Delta - (21) \circ \Delta = \hbar\tilde{\delta})$  by its torsion ideal. Here the additional generator  $\tilde{\delta}$  has bidegree  $(1, 2)$ . If  $\mathcal{S}_{\hbar}$  is the category of topologically free  $\mathbb{K}[[\hbar]]$ -modules, then  $\text{Mod}_{\mathcal{S}_{\hbar}}(\underline{\text{Bialg}}_{\text{qcoco}})$  is the category of quasi-cocommutative, topologically free  $\mathbb{K}[[\hbar]]$ -bialgebras, i.e., such that  $(\Delta - \Delta^{2,1})(A) \subset \hbar A^{\otimes 2}$ .

We define the prop  $\underline{\text{Bialg}}_{\text{cP}}$  of co-Poisson bialgebras as the quotient of  $\underline{\text{Bialg}}\langle\delta\rangle$  ( $\delta$  has bidegree  $(1, 2)$ ) by the relations

$$\Delta = (21) \circ \Delta, \quad \delta + (21) \circ \delta = 0, \quad ((123) + (231) + (312)) \circ (\delta \otimes \text{id}) \circ \delta = 0,$$

$$(\Delta \otimes \text{id}) \circ \delta = ((123) + (213)) \circ (\text{id} \otimes \delta) \circ \Delta,$$

$$\delta \circ m = (m \otimes m) \circ (1324) \circ (\delta \otimes \Delta + \Delta \otimes \delta),$$

$$\delta \circ \eta = 0, \quad (\epsilon \otimes \text{id}) \circ \delta = 0.$$

**Lemma 4.1.** *There is a unique prop morphism  $\underline{\text{SC}} : \underline{\text{Bialg}}_{\text{cP}} \rightarrow \underline{\text{Bialg}}_{\text{qcoco}}/(\hbar)$ , taking  $m, \Delta, \eta, \epsilon$  to their analogues and  $\tilde{\delta}$  to  $\delta$ .  $\underline{\text{SC}}$  is surjective.*

*Proof.* The proof of the first statement is a propic translation of the proof of the following fact: if  $A$  is a quasi-cocommutative topologically free  $\mathbb{K}[[\hbar]]$ -bialgebra, then  $A/\hbar A$ , equipped with  $\delta := (\frac{\Delta - \Delta^{2,1}}{\hbar} \bmod \hbar)$ , is a co-Poisson bialgebra. Since all generators of  $\underline{\text{Bialg}}_{\text{qcoco}}/(\hbar)$  are in the image of  $\underline{\text{SC}}$ ,  $\underline{\text{SC}}$  is surjective.  $\square$

**4.2. Completions.** We denote by  $\underline{\text{Bialg}}_{\text{coco}}$  the quotient of  $\underline{\text{Bialg}}$  by the relation  $\Delta = (21) \circ \Delta$ . Let  $I_n$  be the ideal of  $\underline{\text{Bialg}}_{\text{coco}}$  generated by the  $(\text{id} - \eta \circ \epsilon)^{\otimes p} \circ \Delta^{(p)}$ ,  $p \geq n$ . Here  $\Delta^{(n)} = (\Delta \otimes \text{id}^{\otimes n-2}) \circ \dots \circ \Delta$ . We denote by  $\underline{\text{UE}}$  the completion of  $\underline{\text{Bialg}}_{\text{coco}}$  with respect to the family of ideals  $I_n$ .

We denote by  $J_n$  the ideal of  $\underline{\text{Bialg}}_{\text{cP}}$  with the same generators, and by  $\underline{\text{UE}}_{\text{cP}}$  the completion of  $\underline{\text{Bialg}}_{\text{cP}}$  with respect to the family of ideals  $J_n$ .

We denote by  $K_n$  the ideal of  $\underline{\text{Bialg}}_{\text{qcoco}}$  with the same generators, and by  $\underline{\text{QUE}}$  the completion of  $\underline{\text{Bialg}}_{\text{qcoco}}$  with respect to the family  $K_n$ .

Then  $\underline{\text{SC}}(J_n)$  is contained in the image of  $K_n$  under  $\underline{\text{QUE}} \rightarrow \underline{\text{QUE}}/(\hbar)$ . Therefore:

**Lemma 4.2.** *There is a unique prop morphism  $\underline{\text{SC}} : \underline{\text{UE}}_{\text{cP}} \rightarrow \underline{\text{QUE}}/(\hbar)$ , induced by  $\underline{\text{SC}} : \underline{\text{Bialg}}_{\text{cP}} \rightarrow \underline{\text{Bialg}}_{\text{qcoco}}/(\hbar)$ , which is also surjective.*

**4.3. The isomorphism result.** In [EK], it is shown that there exists a prop morphism  $\underline{Q} : \underline{\text{QUE}} \rightarrow \underline{\text{UE}}_{\text{cP}}[[\hbar]]$ , such that  $(\underline{Q} \bmod \hbar) \circ \underline{\text{SC}} = \text{id}$  (i.e.,  $\underline{Q}$  is a quantization morphism). This implies that  $\underline{\text{SC}} : \underline{\text{UE}}_{\text{cP}} \rightarrow \underline{\text{QUE}}/(\hbar)$  is injective. Then Lemma 4.2 implies that  $\underline{\text{SC}}$  is an isomorphism.

If now  $\underline{Q}'$  is any quantization morphism, then it is a prop morphism  $\underline{Q}' : \underline{\text{QUE}} \rightarrow \underline{\text{UE}}_{\text{cP}}[[\hbar]]$ , such that  $(\underline{Q}' \bmod \hbar)$  is an isomorphism. Applying Hensel's Lemma to the set of all  $\underline{Q}'(p, q) : \underline{\text{QUE}}(p, q) \rightarrow \underline{\text{UE}}_{\text{cP}}(p, q)[[\hbar]]$ , we get that each  $\underline{Q}'(p, q)$  is an isomorphism.

**Proposition 4.1.** *Each quantization morphism  $\underline{Q} : \underline{\text{QUE}} \rightarrow \underline{\text{UE}}_{\text{cP}}[[\hbar]]$  is an isomorphism. So the set of all quantization morphisms is a torsor over the groups  $\text{Aut}_1(\underline{\text{QUE}})$  acting on the right and  $\text{Aut}_1(\underline{\text{UE}}_{\text{cP}}[[\hbar]])$  acting on the left.*

**4.4. Modules over topological props.** Let  $\mathcal{S}$  be the category of vector spaces and let us describe the category  $\text{Mod}_{\mathcal{S}}(\underline{\text{UE}})$ .

**Lemma 4.3.**  *$\text{Mod}_{\mathcal{S}}(\underline{\text{UE}})$  is the category of universal enveloping algebras over  $\mathbb{K}$ , so it is equivalent to the category of Lie algebras.*

*Proof.* We have a morphism  $\underline{\text{Bialg}}_{\text{coco}} \rightarrow \underline{\text{UE}}$ , so if  $A$  is a  $\underline{\text{UE}}$ -module, then it is a cocommutative bialgebra  $(A, m_A, \Delta_A, \eta_A, \epsilon_A)$ . The condition that  $A$  is a  $\underline{\text{UE}}$ -module means that for  $x_n \in I_n$  and  $a \in \oplus_p A^{\otimes p}$ ,  $\rho(x_n)(a)$  should tend to zero as  $n \rightarrow \infty$ . Since the topology of  $\oplus_p A^{\otimes p}$  is discrete, this means that this sequence vanishes for  $n$  large enough. In particular, for  $a \in A$ , the sequence  $(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n} \circ \Delta_A^{(n)}(a)$  vanishes for large  $n$ . One checks that this condition is actually equivalent to  $A$  being a  $\underline{\text{UE}}$ -module. The Milnor-Moore theorem ([MM]) then says that  $A$  is a universal enveloping algebra.  $\square$

It follows that  $\text{Mod}_{\mathcal{S}}(\underline{\text{UE}}_{\text{CP}})$  is the category of universal enveloping algebras with a co-Poisson structure, and is equivalent to the category of Lie bialgebras over  $\mathbb{K}$ .

Recall now that  $\mathcal{S}_{\hbar}$  is the category of topologically free  $\mathbb{K}[[\hbar]]$ -modules.

**Lemma 4.4.**  *$\text{Mod}_{\mathcal{S}_{\hbar}}(\underline{\text{UE}}_{\text{CP}}[[\hbar]])$  is equivalent to the category of topologically free Lie bialgebras over  $\mathbb{K}[[\hbar]]$  (i.e., Lie bialgebras in the category  $\mathcal{S}_{\hbar}$ ).*

*Proof.* The same argument as above shows that the objects of  $\text{Mod}_{\mathcal{S}_{\hbar}}(\underline{\text{UE}}[[\hbar]])$  are the topologically free  $\mathbb{K}[[\hbar]]$ -bialgebras, such that for  $a \in A$ , the  $\hbar$ -adic valuation of  $(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n} \circ \Delta_A^{(n)}(a)$  tends to zero as  $n \rightarrow \infty$ . A topological version of the Milnor-Moore theorem then says that  $A$  is the topological enveloping algebra of a Lie algebra over  $\mathbb{K}[[\hbar]]$ , which is a topologically free  $\mathbb{K}[[\hbar]]$ -module.  $\square$

We now study  $\text{Mod}_{\mathcal{S}_{\hbar}}(\underline{\text{QUE}})$ .

**Proposition 4.2.** *The category  $\text{Mod}_{\mathcal{S}_{\hbar}}(\underline{\text{QUE}})$  identifies with the category  $\text{QUE}$  of QUE-algebras over  $\mathbb{K}$ .*

*Proof.* Let  $A$  be a module over  $\text{QUE}$  in the category  $\mathcal{S}_{\hbar}$ . We have a prop morphism  $\underline{\text{Bialg}}_{\text{qco}} \rightarrow \underline{\text{QUE}}$ , so  $A$  is a quasi-cocommutative bialgebra. As above, the condition that  $A$  is a  $\underline{\text{QUE}}$ -module is equivalent to the condition that for each  $a \in A$ , the  $\hbar$ -adic valuation of  $(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n} \circ \Delta_A^{(n)}(a)$  tends to infinity when  $n \rightarrow \infty$ . Let  $A_0 = A/\hbar A$ . Then this condition implies that for each  $a_0 \in A_0$ ,  $(\text{id} - \eta_{A_0} \circ \epsilon_{A_0})^{\otimes n} \circ \Delta_{A_0}^{(n)}$  vanishes for  $n$  large enough. Therefore  $A_0$  is a universal enveloping algebra. Let us show that  $A$  is a Hopf algebra: the antipode of  $A$  is given by the formula

$$S(a) = \epsilon(a)1 - a_0 + (a^{(1)})_0(a^{(2)})_0 - (a^{(1)})_0(a^{(2)})_0(a^{(3)})_0 \dots,$$

(we set  $x_0 = x - \epsilon_A(x)1$ ) i.e.,  $S = \sum_{n \geq 0} (-1)^n m_A^{(n)} \circ (\text{id} - \eta_A \circ \epsilon_A)^{\otimes n} \circ \Delta_A^{(n)}$ , where  $m_A^{(n)}$  is the  $n$ fold product of  $A$ . Therefore  $A$  is a  $\mathbb{K}[[\hbar]]$ -Hopf algebra, whose reduction modulo  $\hbar$  is a universal enveloping algebra, so it is a QUE algebra.

Conversely, let us show that any QUE algebra  $A$  is a  $\underline{\text{QUE}}$ -module. We should show that for any  $a \in A$ , the  $\hbar$ -adic valuation of  $(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n} \circ \Delta_A^{(n)}$  tends to infinity with  $n$ . Let  $A_0 = A/\hbar A$ . By assumption on  $A_0$ , there exists an integer  $n_1$  such that  $(\text{id} - \eta_{A_0} \circ \epsilon_{A_0})^{\otimes n_1} \circ \Delta_{A_0}^{(n_1)}(a \bmod \hbar) = 0$ , so  $(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n_1} \circ \Delta_A^{(n_1)}(a) \in \hbar A^{\otimes n_1}$ . Let us denote by  $a_1$  the class of  $\frac{1}{\hbar}(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n_1} \circ \Delta_A^{(n_1)}(a)$  modulo  $\hbar$ . This is an element of  $A_0^{\otimes n_1}$ , so there exists  $n_2$  such that

$$((\text{id} - \eta_A \circ \epsilon_A)^{\otimes n_2} \circ \Delta^{(n_2)}) \otimes \text{id}^{\otimes n_1 - 1}(a_1) = 0,$$

therefore  $(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n_1+n_2} \circ \Delta_A^{(n_1+n_2)}(a)$  belongs to  $\hbar^2 A^{\otimes n_1+n_2}$ . In the same way, one constructs a sequence of integers  $(n_k)_{k \geq 1}$ , such that  $(\text{id} - \eta_A \circ \epsilon_A)^{\otimes n_1+\dots+n_k} \circ \Delta_A^{(n_1+\dots+n_k)}$  belongs to  $\hbar^k A^{\otimes n_1+\dots+n_k}$ . This implies that  $A$  is a QUE-module.  $\square$

Proposition 4.1 now implies:

**Theorem 4.1.** (see [EK2]) *Each quantization morphism induces an equivalence of categories between (a) the category QUE of QUE-algebras over  $\mathbb{K}$ , and (b) the category  $\text{LBA}_\hbar$  of topologically free  $\mathbb{K}[[\hbar]]$ -Lie bialgebras.*

One can define the prop Hopf of Hopf algebras as  $\text{Hopf} = \text{Bialg}(S)/(\text{relations})$ , where  $S$  has bidegree  $(1,1)$  and the relations express the axioms for the antipode. Then we have a prop morphism  $\text{Hopf} \rightarrow \text{QUE}$ , taking  $S$  to  $\sum_{n \geq 0} (-1)^n m^{(n)} \circ (\text{id} - \eta \circ \epsilon)^{\otimes n} \circ \Delta^{(n)}$ . We have  $S^2 \in \text{id} + \hbar \text{QUE}(1,1)$ , so we have a 1-parameter subgroup of  $\text{QUE}(1,1)^\times$ ,  $\lambda \mapsto (S^2)^\lambda$ , generated by  $\log(S^2) \in \text{QUE}(1,1)$ .

**Proposition 4.3.** (see [EK3], Proposition A3) *Any quantization morphism  $\text{QUE} \rightarrow \text{UE}_{\text{CP}}[[\hbar]]$  takes  $\log(S^2)$  to a multiple of  $\mu \circ \delta$ .*

*Proof.* If  $P$  is a prop, let us say that a prop automorphism of  $P$  is  $\theta \in P(1,1)$ , such that  $x \circ \theta^{\otimes p} = \theta^{\otimes q}$  for any  $x \in P(p,q)$ . A prop derivation is the corresponding infinitesimal object. Then  $S^2$  is a prop automorphism of QUE, so if  $Q$  is a quantization functor,  $Q(S^2)$  is a prop automorphism of  $\text{UE}_{\text{CP}}[[\hbar]]$ . In particular, it commutes with the idempotents  $p_n$  (see Lemma 6.3), so it induces a prop automorphism of  $\text{LBA}[[\hbar]]$ . Then  $Q(\log(S^2))$  is a prop derivation of  $\text{LBA}[[\hbar]]$ . In [Enr], we have shown that any such derivation is proportional to  $\mu \circ \delta$  (here  $\mu, \delta$  are the generators of  $\text{LBA}$ ). This derivation of  $\text{LBA}[[\hbar]]$  extends uniquely to a derivation of  $\text{UE}_{\text{CP}}[[\hbar]]$ , also given by the formula  $\mu \circ \delta$  (here  $\mu, \delta$  are generators of  $\text{UE}_{\text{CP}}$ ).  $\square$

This proposition was proved in [EK3] when  $Q$  is an Etingof-Kazhdan quantization morphism.

## 5. DEQUANTIZATION OF QTQUE ALGEBRAS

**5.1. Props of some quasitriangular structures.** Recall that the prop  $\text{Bialg}_{\text{coco}}$  is the quotient of the prop Bialg by the ideal generated by  $\Delta = (21) \circ \Delta$ : it is the prop of cocommutative bialgebras.

Define  $\text{Bialg}_{\text{coco,qt}}$  as  $\text{Bialg}_{\text{coco}}\langle r \rangle / (\text{relations})$ , where  $r$  has bidegree  $(0,2)$  and the relations are:

$$\begin{aligned} (\Delta \otimes \text{id}) \circ r &= r^{1,3} \eta^2 + \eta^1 r^{2,3}, \quad (\text{id} \otimes \Delta) \circ r = r^{1,3} \eta^2 + r^{1,2} \eta^3, \\ (m \otimes m) \circ (1324) \circ (t \otimes \Delta) &= (m \otimes m) \circ (1324) \circ (\Delta \otimes t) \end{aligned}$$

(here  $t = r + (21) \circ r$ ), together with the analogue of (2). The  $\text{Mod}_{\mathcal{S}}(\text{Bialg}_{\text{coco,qt}})$  is the category of pairs  $(A, r_A)$ , where  $A$  is a cocommutative bialgebra, and  $r_A \in A^{\otimes 2}$  is such that  $(\Delta_A \otimes \text{id})(r_A) = r_A^{1,3} + r_A^{2,3}$ ,  $(\text{id} \otimes \Delta_A)(r_A) = r_A^{1,2} + r_A^{1,3}$ , the identity  $[t_A, \Delta_A(x)] = 0$  holds for any  $x \in A$ , where  $t_A = r_A + r_A^{2,1}$ , and  $\text{CYB}(r_A) = 0$  (the two first conditions mean that  $r_A \in \text{Prim}(A)^{\otimes 2}$ ). Such a pair  $(A, r_A)$  gives rise to a co-Poisson cocommutative bialgebra, with  $\delta_A(x) = [r_A, \Delta_A(x)]$ ; this corresponds to a prop morphism  $\text{Bialg}_{\text{CP}} \rightarrow \text{Bialg}_{\text{coco,qt}}$ . We have an obvious prop morphism  $\text{CYBA} \rightarrow \text{Bialg}_{\text{coco,qt}}$  (see Section 3).

Define  $\text{Bialg}_{\text{QT}}$  as  $\text{Bialg}\langle R, R^{-1} \rangle / (\text{relations})$ , where  $R, R^{-1}$  have bidegree  $(0,2)$  and the relations are

$$\begin{aligned} (\Delta \otimes \text{id}) \circ R &= (\text{id} \otimes \text{id} \otimes m)(R^{1,3} R^{2,4}), \quad (\text{id} \otimes \Delta) \circ R = (m \otimes \text{id} \otimes \text{id})(R^{1,4} R^{2,3}), \\ (m \otimes m) \circ (2314) \circ (\Delta \otimes R) &= (m \otimes m) \circ (1324) \circ (R \otimes \Delta), \\ (m \otimes m) \circ (1324) \circ (R \otimes R^{-1}) &= \eta^{\otimes 2}. \end{aligned}$$



Then  $\text{Mod}_S(\underline{\text{Bialg}}_{\text{QT}})$  is the category of quasitriangular bialgebras, i.e., pairs  $(A, R_A)$ , where  $A$  is a bialgebra and  $R_A \in A^{\otimes 2}$  is invertible, such that  $(\Delta_A \otimes \text{id})(R_A) = R_A^{1,3} R_A^{2,3}$ ,  $(\text{id} \otimes \Delta_A)(R_A) = R_A^{1,3} R_A^{1,2}$ , and  $\Delta_A^{2,1}(x) R_A = R_A \Delta(x)$  holds for any  $x \in A$ . Then  $(A, R_A)$  is a solution of the QYBE.

We define now  $\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}$  as the quotient of  $\underline{\text{Bialg}}_{\text{QT}}[[\hbar]] \langle \tilde{\delta}, \tilde{R} \rangle / (\Delta - (21) \circ \Delta = \hbar \tilde{\delta}, R = \eta^{\otimes 2} + \hbar \tilde{R})$  by its torsion ideal. Then  $\text{Mod}_{S_\hbar}(\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}})$  is the category of quasitriangular quasi-cocommutative  $\mathbb{K}[[\hbar]]$ -bialgebras  $(A, R_A)$ , such that  $R_A \in 1 + \hbar A^{\otimes 2}$ . We have a prop morphism  $\underline{\text{QYBA}} \rightarrow \underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}$ , taking  $m, \eta$  to their analogues and  $\rho$  to  $\tilde{R}$  (see Section 3).

**Lemma 5.1.** *There exists a unique prop morphism  $\underline{\text{SC}} : \underline{\text{Bialg}}_{\text{coco}, \text{qt}} \rightarrow \underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}/(\hbar)$ , taking  $m, \Delta, \eta, \epsilon$  to the reductions modulo  $\hbar$  of their analogues, and taking  $r$  to the reduction modulo  $\hbar$  of  $R$ .  $\underline{\text{SC}}$  is surjective.*

*Proof.* The proof that this assignment on generators defines a morphism of props is a propic version of the proof of the following fact, due to Drinfeld: if  $(A, R_A)$  is a quasi-cocommutative quasitriangular bialgebra, such that  $R_A \in 1 + \hbar A^{\otimes 2}$ , and if  $A_0 = A/\hbar A$ ,  $r_A = (\frac{R_A - 1}{\hbar} \bmod \hbar)$ , then  $(A_0, r_A)$  is as above. Let us recall the proof of this fact. The identities  $(\Delta_A \otimes \text{id})(R_A) = R_A^{1,3} R_A^{2,3}$  and  $(\text{id} \otimes \Delta_A)(R_A) = R_A^{1,3} R_A^{1,2}$  imply, after we subtract 1, divide by  $\hbar$  and reduce modulo  $\hbar$ , that  $(\Delta_{A_0} \otimes \text{id})(r_A) = r_A^{1,3} + r_A^{2,3}$  and  $(\text{id} \otimes \Delta_{A_0})(r_A) = r_A^{1,2} + r_A^{1,3}$  (in the propic proof, dividing by  $\hbar$  is possible because  $\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}$  is constructed to be torsion-free).  $R_A$  satisfies the QYBE, so subtracting from this identity  $R_A^{1,2} + R_A^{1,3} + R_A^{2,3} - 2$ , dividing by  $\hbar^2$  and reducing modulo  $\hbar$ , we find that  $r_A$  satisfies the CYBE (again, the propic version uses that  $\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}$  is torsion-free). Finally, we have the identity  $(R_A^{2,1} R_A) \Delta_A(x) = \Delta_A(x) (R_A^{2,1} R_A)$  for any  $x \in A$ . Subtracting  $\Delta_A(x)$  from both sides, dividing by  $\hbar$  and reducing modulo  $\hbar$ , we find that  $[r_A + r_A^{2,1}, \Delta_{A_0}(x)] = 0$  for any  $x \in A_0$  (in the propic case, we use the torsion-freeness of  $\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}$  once more). All the generators of  $\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}/(\hbar)$  are in the image of  $\underline{\text{SC}}$ , so  $\underline{\text{SC}}$  is surjective.  $\square$

**5.2. Completions.** We denote by  $I'_n$  the ideal of  $\underline{\text{Bialg}}_{\text{coco}, \text{qt}}$  generated by the  $(\text{id} - \eta \circ \epsilon)^{\otimes p} \circ \Delta^{(p)}$ ,  $p \geq n$ , and  $\underline{\text{UE}}_{\text{qt}}$  to be the completion of  $\underline{\text{Bialg}}_{\text{coco}, \text{qt}}$  with respect to the family  $I'_n, n \geq 0$ .

We denote by  $J'_n$  the ideal of  $\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}$  generated by the analogous elements, and by  $\underline{\text{QUE}}_{\text{QT}}$  the completion of  $\underline{\text{Bialg}}_{\text{qcoco}, \text{QT}}$  with respect to the family  $J'_n, n \geq 0$ . Similarly to Lemma 4.2, we have:

**Lemma 5.2.**  *$\underline{\text{SC}}$  extends continuously to a unique morphism of props  $\underline{\text{SC}} : \underline{\text{UE}}_{\text{coco}, \text{qt}} \rightarrow \underline{\text{QUE}}_{\text{QT}}/(\hbar)$ , which is also surjective.*

**5.3. The isomorphism result.** In [EK2], it is shown that there exists a prop morphism  $\underline{Q} : \underline{\text{QUE}}_{\text{QT}} \rightarrow \underline{\text{UE}}_{\text{qt}}[[\hbar]]$ , such that  $(\underline{Q} \bmod \hbar) \circ \underline{\text{SC}}$  is the identity (i.e.,  $\underline{Q}$  is a quantization morphism). This implies that  $\underline{\text{SC}}$  is injective. Together with Lemma 5.2, this implies that  $\underline{\text{SC}}$  is an isomorphism. Now Hensel's lemma implies that any quantization functor  $\underline{Q}$  is a prop isomorphism. We have proved:

**Proposition 5.1.** *Any quantization morphism  $\underline{Q} : \underline{\text{QUE}}_{\text{QT}} \rightarrow \underline{\text{UE}}_{\text{qt}}[[\hbar]]$  is a prop isomorphism.*

**5.4. Modules over quasitriangular props.**

**Lemma 5.3.**  *$\text{Mod}_S(\underline{\text{UE}}_{\text{qt}})$  is equivalent to the category of quasitriangular Lie bialgebras over  $\mathbb{K}$ , i.e., pairs  $(\mathfrak{a}, r_{\mathfrak{a}})$  of a Lie algebra  $\mathfrak{a}$  over  $\mathbb{K}$  and  $r_{\mathfrak{a}} \in \mathfrak{a}^{\otimes 2}$ , such that  $\text{CYB}(r_{\mathfrak{a}}) = 0$  and  $r_{\mathfrak{a}} + r_{\mathfrak{a}}^{2,1}$  is invariant.*

*Proof.* Let  $(A, r_A)$  be a module over  $\underline{\text{UE}}_{\text{qt}}$ . Then  $A$  is a  $\underline{\text{UE}}$ -module, so it is a universal enveloping algebra. Let  $\mathfrak{a} = \text{Prim}(A)$ . We know that  $r_A \in \mathfrak{a}^{\otimes 2}$ ,  $r_A$  satisfies the CYBE and  $t_A := r_A + r_A^{2,1}$  commutes with the image of  $\Delta : A \rightarrow A^{\otimes 2}$ ; therefore  $[t_A, x^1 + x^2] = 0$  for any  $x \in \mathfrak{a}$ . Conversely, since  $\mathfrak{a}$  generates  $A$  as an algebra, this last condition implies that  $t_A$  commutes with the image of  $\Delta_A$ .  $\square$

In the same way, one shows that  $\text{Mod}_{S_{\hbar}}(\underline{\text{UE}}_{\text{qt}}[[\hbar]])$  is equivalent to the category  $\text{LBA}_{\text{qt}, \hbar}$  of quasitriangular Lie bialgebras in the category of topologically free  $\mathbb{K}[[\hbar]]$ -modules, i.e., of pairs  $(\mathfrak{a}, r_{\mathfrak{a}})$ , where  $\mathfrak{a}$  is a topologically free  $\mathbb{K}[[\hbar]]$ -module and  $r_{\mathfrak{a}} \in \mathfrak{a}^{\otimes 2}$  is a solution of CYBE, such that  $r_{\mathfrak{a}} + r_{\mathfrak{a}}^{2,1}$  is invariant.

**Lemma 5.4.**  *$\text{Mod}_{S_{\hbar}}(\underline{\text{QUE}})$  is equivalent to the category  $\text{QUE}_{\text{QT}}$  of quasitriangular QUE algebras over  $\mathbb{K}$ , i.e., pairs  $(A, R_A)$  of a QUE algebra over  $\mathbb{K}$  and  $R_A \in 1 + \hbar A^{\otimes 2}$ , such that  $(A, R_A)$  is a quasitriangular bialgebra.*

These lemmas and Proposition 5.1 imply:

**Theorem 5.1.** *Each quantization morphism  $\underline{Q} : \underline{\text{QUE}}_{\text{QT}} \rightarrow \underline{\text{UE}}_{\text{qt}}[[\hbar]]$  gives rise to an equivalence of categories between  $\text{QUE}_{\text{QT}}$  and  $\text{LBA}_{\text{qt}, \hbar}$ .*

Recall that the quantization morphisms from [EK] do not alter the algebra structure of  $U(\mathfrak{a})$ , when  $\mathfrak{a}$  is quasitriangular. When  $\underline{Q}$  is such a quantization functor, Theorem 5.1 can be made more precise as follows:

**Theorem 5.2.** *Let  $\mathfrak{a}_0$  be a Lie algebra over  $\mathbb{K}$  and set  $\mathfrak{a} = \mathfrak{a}_0[[\hbar]]$ . Then each quantization morphism from [EK] sets up a bijection between the following coset spaces:*

- (a) *the set of  $r_{\mathfrak{a}} \in \mathfrak{a}^{\otimes 2}$ , such that  $r_{\mathfrak{a}} + r_{\mathfrak{a}}^{2,1}$  is invariant and  $\text{CYB}(r_{\mathfrak{a}}) = 0$ , modulo the action of  $\text{Aut}_1(\mathfrak{a})$ ;*
- (b) *the set of quasitriangular QUE algebra structures  $(\Delta_{\mathfrak{a}}, R_{\mathfrak{a}})$  on  $U(\mathfrak{a})$ , modulo the action of  $\text{Aut}_1(U(\mathfrak{a}))$ .*

*Here  $\text{Aut}_1(\mathfrak{a})$  (resp.,  $\text{Aut}_1(U(\mathfrak{a}))$ ) is the group of Lie algebra (resp., algebra) automorphisms of  $\mathfrak{a}$  (resp.,  $U(\mathfrak{a})$ ), whose reduction modulo  $\hbar$  is the identity.*

*Proof.* The proof is based on the following facts: the group  $\text{Aut}_1(U(\mathfrak{a}))$  acts transitively on  $\{\text{cocommutative bialgebra structures on } (U(\mathfrak{a}), m_0) \text{ deforming } \Delta_0\}$ , by taking  $(\theta, \Delta)$  to  $\theta^{\otimes 2} \circ \Delta \circ \theta^{-1}$ . The isotropy subgroup of  $\Delta_0$  is  $\text{Aut}_1(\mathfrak{a})$ . Here  $(m_0, \Delta_0)$  are the undeformed structure maps of  $U(\mathfrak{a})$ . These facts are proved using co-Hochschild cohomology.  $\square$

This correspondence is such that  $R_{\mathfrak{a}} = 1 + \hbar r_{\mathfrak{a}} + O(\hbar^2)$  and the map  $r_{\mathfrak{a}} \mapsto R_{\mathfrak{a}}$  is expressed by the same universal formulas as in Section 3.

5.5. We will derive from this a classification of twistors related to a given associator.

Let  $\mathfrak{a}_0$  be a Lie algebra over  $\mathbb{K}$ . Let  $\mathfrak{a} := \mathfrak{a}_0[[\hbar]]$  and  $t_{\mathfrak{a}} \in S^2(\mathfrak{a})^{\mathfrak{a}}$  be a symmetric invariant tensor. Let  $\Phi$  be a Drinfeld associator. We denote the specialization of  $\Phi$  to  $(\mathfrak{a}, t_{\mathfrak{a}})$  by  $\Phi_{\mathfrak{a}}$ .

If  $J \in 1 + \hbar U(\mathfrak{a})^{\otimes 2}$ , we set

$$\tilde{d}(J) := (J^{2,3} J^{1,23})^{-1} J^{1,2} J^{12,3}.$$

In this section, we describe the set  $X$  of all  $J \in 1 + \hbar U(\mathfrak{a})^{\otimes 2}$ , such that  $\tilde{d}(J) = \Phi_{\mathfrak{a}}$ . We denote by  $(u, J) \mapsto u * J$  the action of  $1 + \hbar U(\mathfrak{a})$  on  $1 + \hbar U(\mathfrak{a})^{\otimes 2}$  defined by  $u * J := u^1 u^2 J(u^{12})^{-1}$ . If  $\tilde{d}(J)$  is invariant, then  $\tilde{d}(u * J) = \tilde{d}(J)$ .

We set  $Y = \{\rho \in \mathfrak{a}^{\otimes 2} \mid \text{CYB}(\rho) = 0, \rho + \rho^{2,1} = t_{\mathfrak{a}}\}$ .

In [EK, EK2, Enr2], we constructed a map  $\rho \mapsto J_{\Phi}(\rho)$ , such that if  $\rho$  satisfies the CYBE, then  $\tilde{d}(J_{\Phi}(\rho)) = \Phi(\hbar \tau^{1,2}, \hbar \tau^{2,3})$ ; here  $\tau = \rho + \rho^{2,1}$ . The assignment  $\rho \mapsto J_{\Phi}(\rho)$  defines a map from  $Y$  to  $X$ .

Recall that  $\exp(\hbar\mathfrak{a})$  is a multiplicative subgroup of  $1 + \hbar U(\mathfrak{a})$ . It acts on  $\{\rho \in \mathfrak{a}^{\otimes 2} | \rho + \rho^{2,1} = t_{\mathfrak{a}} \text{ and } \text{CYB}(\rho) = 0\}$  by conjugation.

**Theorem 5.3.** *Let  $J$  be an element of  $X$ . Then there exists  $u \in 1 + \hbar U(\mathfrak{a})$  and  $\rho \in Y$ , such that  $J = u * J_{\Phi}(\rho)$ . Two pairs  $(u, \rho)$  and  $(u', \rho')$  determine the same  $J$  iff there exists an element  $v \in \exp(\hbar\mathfrak{a})$ , such that  $u' = uv$  and  $J' = v^{-1} * J$ . In other words,  $(u, \rho) \mapsto u * J_{\Phi}(\rho)$  defines a bijection*

$$(1 + \hbar U(\mathfrak{a})) \times_{\exp(\hbar\mathfrak{a})} Y \xrightarrow{\sim} X.$$

*Proof.* Recall that the Lie algebra  $\text{Der}(\mathfrak{a}, t_{\mathfrak{a}})$  of derivations of  $\mathfrak{a}$  leaving  $t_{\mathfrak{a}}$  invariant, acts on  $Y$ .

Let  $J$  belong to  $X$ . We will prove the following statement. There exist sequences  $\rho_n \in Y$ ,  $\kappa_n \in U(\mathfrak{a})$ ,  $\gamma_n \in \text{Der}(\mathfrak{a}, t_{\mathfrak{a}})$ , and algebra automorphisms  $\theta_n \in \text{Aut}_1(U(\mathfrak{a}))$ , such that:

- (1)  $J_0 = J$ ,  $J_n - J_{\Phi}(\rho_n) = O(\hbar^n)$ ,  $\rho_{n+1} = \exp(\hbar^{n+1}\gamma_{n+1})^{\otimes 2}(\rho_n)$ ,  $J_{n+1} = (1 + \hbar^{n+1}\kappa_{n+1}) * J_n$ ,  $\theta_n = \text{id} + O(\hbar^n)$ ;
- (2)  $\theta_n$  sets up an isomorphism between the quasitriangular QUE algebras

$$(U(\mathfrak{a}), m_0, \text{Ad}(J_n) \circ \Delta_0, J_n^{2,1} e^{\hbar t_{\mathfrak{a}}/2} J_n^{-1}). \quad (3)$$

and

$$(U(\mathfrak{a}), m_0, \text{Ad}(J_{\Phi}(\rho_n)) \circ \Delta_0, J_{\Phi}(\rho_n)^{2,1} e^{\hbar t_{\mathfrak{a}}/2} J_{\Phi}(\rho_n)^{-1}). \quad (4)$$

We first define  $\rho_0$ . Twisting the quasitriangular quasi-Hopf algebra  $(U(\mathfrak{a}), m_0, \Delta_0, e^{\hbar t_{\mathfrak{a}}/2}, \Phi_{\mathfrak{a}})$  by  $J$ , we obtain a quasitriangular QUE algebra

$$(U(\mathfrak{a}), m_0, \text{Ad}(J) \circ \Delta_0, J^{2,1} e^{\hbar t_{\mathfrak{a}}/2} J^{-1}).$$

According to Theorem 5.2, there exists  $\rho_0 \in Y$ , such that this algebra is isomorphic to

$$(U(\mathfrak{a}), m_0, \text{Ad}(J_{\Phi}(\rho_0)) \circ \Delta_0, J_{\Phi}(\rho_0)^{2,1} e^{\hbar t_{\mathfrak{a}}/2} J_{\Phi}(\rho_0)^{-1}).$$

We will construct these sequences inductively (the base of induction is obvious). We will write  $J, \rho, J', \rho', \kappa, \theta, \theta''$  instead of  $J_n, \rho_n, J_{n+1}, \rho_{n+1}, \kappa_{n+1}, \theta_n, \theta_{n+1}$ .

Since the multiplication is the same in algebras (3) and (4), we have  $\theta \in \text{Aut}_1(U(\mathfrak{a}))$ . Moreover,

$$\text{Ad}(J) \circ \Delta_0 = \theta^{\otimes 2} \circ \text{Ad}(J_{\Phi}(\rho)) \circ \Delta_0 \circ \theta^{-1}, \quad J^{2,1} e^{\hbar t_{\mathfrak{a}}/2} J^{-1} = \theta^{\otimes 2} (J_{\Phi}(\rho)^{2,1} e^{\hbar t_{\mathfrak{a}}/2} J_{\Phi}(\rho)^{-1}). \quad (5)$$

By hypothesis, we have  $J - J_{\Phi}(\rho) = O(\hbar^n)$  and  $\theta = \text{id} + O(\hbar^n)$ . Let  $K \in U(\mathfrak{a}_0)^{\otimes 2}$  and  $\gamma \in \text{Der}(\mathfrak{a}_0, U(\mathfrak{a}_0)^{\otimes 2})$  be the reductions modulo  $\hbar$  of  $\hbar^{-n}(J - J_{\Phi}(\rho))$  and  $\hbar^{-n}(\theta - \text{id})$ . Then (5) imply

$$[K, \Delta_0(x)] = (\gamma \otimes \text{id} + \text{id} \otimes \gamma)(\Delta_0(x)) - \Delta_0(\gamma(x)) \quad (6)$$

for any  $x \in U(\mathfrak{a}_0)$ , and  $K = K^{2,1}$ . Moreover, since  $\tilde{d}(J) = \tilde{d}(J_{\Phi}(\rho))$ , we have  $d(K) := K^{12,3} - K^{1,23} - K^{2,3} + K^{1,2} = 0$ .

The equations in  $K$  imply that there exists  $\kappa \in U(\mathfrak{a}_0)$ , such that  $K = \kappa^1 + \kappa^2 - \kappa^{12} =: d(\kappa)$ . Then (6) implies that for  $x \in \mathfrak{a}_0$ ,  $\gamma(x) - [\kappa, x] \in \mathfrak{a}_0$ . Therefore  $\gamma = \text{ad}(\kappa) + \gamma_0$ , where  $\gamma_0 \in \text{Der}(U(\mathfrak{a}_0))$  is induced by a derivation of  $U(\mathfrak{a}_0)$ .

We now view  $\kappa$  as an element of  $U(\mathfrak{a})$  and set  $J' := (1 + \hbar^n \kappa)^{-1} * J$ . Then  $J' = J - \hbar^n K + O(\hbar^{n+1})$ , therefore  $J' - J_{\Phi}(\rho') = O(\hbar^{n+1})$ . Set  $\theta' = \text{Ad}(1 + \hbar^n \kappa)^{-1} \circ \theta$ , then  $\theta' = \text{id} + \hbar^n \gamma_0 + O(\hbar^n)$ , where  $\gamma_0$  is viewed as a derivation of  $U(\mathfrak{a})$ , preserving  $\mathfrak{a}$ .

Now the second equation in (5) implies that  $J' e^{\hbar t_{\mathfrak{a}}}(J')^{-1} = (\theta')^{\otimes 2} (J_{\Phi}(\rho) e^{\hbar t_{\mathfrak{a}}} J_{\Phi}(\rho)^{-1})$ . So  $J' t_{\mathfrak{a}}(J')^{-1} = (\theta')^{\otimes 2} (J_{\Phi}(\rho) t_{\mathfrak{a}} J_{\Phi}(\rho)^{-1})$ . The coefficient of  $\hbar^n$  in this identity yields  $(\gamma_0 \otimes \text{id} + \text{id} \otimes \gamma_0)(t_{\mathfrak{a}}) = 0$ .

Set  $\rho' := \exp(\hbar^n \gamma_0)^{\otimes 2}(\rho)$ . Then  $J_{\Phi}(\rho') = J_{\Phi}(\rho) + O(\hbar^{n+1})$ . Then  $J' - J_{\Phi}(\rho') = O(\hbar^{n+1})$ .

Set  $\theta'' = \theta' \circ \exp(\hbar^n \gamma_0)^{-1}$ . Then

$$\text{Ad}(J') \circ \Delta_0 = \theta''^{\otimes 2} \circ \text{Ad}(J_\Phi(\rho')) \circ \Delta_0 \circ (\theta'')^{-1}, \quad J'^{2,1} e^{\hbar t_a/2} (J')^{-1} = \theta'^{\otimes 2} (J_\Phi(\rho')^{2,1} e^{\hbar t_a/2} J_\Phi(\rho')^{-1}),$$

where the second equation follows from the fact that  $\gamma_0$  leaves  $t_a$  invariant. These equations mean that  $\theta'' = \text{id} + O(\hbar^{n+1})$  is an isomorphism between the quasitriangular QUE algebras

$$(U(\mathfrak{a}), m_0, \text{Ad}(J) \circ \Delta_0, J^{2,1} e^{\hbar t_a/2} J^{-1})$$

and

$$(U(\mathfrak{a}), m_0, \text{Ad}(J_\Phi(\rho)) \circ \Delta_0, J_\Phi(\rho')^{2,1} e^{\hbar t_a/2} J_\Phi(\rho')^{-1}),$$

and we recall that  $J' = J_\Phi(\rho') + O(\hbar^{n+1})$ . This completes the induction step.

The fact that  $u * J_\Phi(\rho) = u' * J_\Phi(\rho')$  implies that  $(u, \rho)$  and  $(u', \rho')$  are related by the action of  $\exp(\hbar a)$  is proved by a co-Hochschild cohomology argument: let  $n$  be the smallest integer such that  $u - u' = O(\hbar^n)$ ,  $\rho - \rho' = O(\hbar^n)$ . Then if  $v, \sigma$  are the reductions modulo  $\hbar$  of  $\hbar^{-n}(u - u')$ ,  $\hbar^{-n}(\rho - \rho')$ , then  $d(u) + \sigma = 0$ , which implies  $\sigma = 0$  and  $u \in \mathfrak{a}_0$  by co-Hochschild cohomology.  $\square$

## 6. STRUCTURE RESULTS FOR SOME PROPS

**6.1. Props constructed from operads.** We define  $\underline{\text{Alg}}, \underline{\text{Alg}}_{\text{comm}}, \underline{\text{Poisson}}$  as the props associated to the operads of associative, commutative and Poisson algebras. Let  $\mathcal{FA}_N, \mathcal{FC}_N$  and  $\mathcal{FP}_N$  be the free associative, commutative and Poisson algebras in  $N$  variables with degrees  $\delta_1, \dots, \delta_N$ . Then  $\mathcal{FP}_N = S(\mathcal{FL}_N)$ , where  $\mathcal{FL}_N$  is the free Lie algebra with  $N$  generators, and  $\mathcal{FA}_N = U(\mathcal{FL}_N)$ .

Then we have:

**Lemma 6.1.** *We have*

$$\underline{\text{Alg}}(N, n) = (\mathcal{FA}_N^{\otimes n})_{\sum_{i=1}^N \delta_i}, \quad \underline{\text{Alg}}_{\text{comm}}(N, n) = (\mathcal{FC}_N^{\otimes n})_{\sum_{i=1}^N \delta_i}, \quad \underline{\text{Poisson}}(N, n) = (\mathcal{FP}_N^{\otimes n})_{\sum_{i=1}^N \delta_i}.$$

Here the subscript means the homogeneous part of degree  $\sum_{i=1}^N \delta_i$ .

*Proof.* The proof is based on the existence of free objects in the categories of associative, Poisson and commutative algebras.  $\square$

We can also define the props  $\underline{\text{Coalg}}, \underline{\text{Coalg}}_{\text{coco}}, \underline{\text{Coalg}}_{\text{cP}}$  of coassociative (resp., cocommutative, co-Poisson) coalgebras, corresponding to the dual cooperads. Then  $\underline{\text{CoX}}(n, N) = \underline{X}(N, n)$ .

We now define  $\underline{\text{Alg}}_{\text{qcom}}$  as the quotient of  $\underline{\text{Alg}}[[\hbar]]\langle \tilde{P} \rangle / (m - (21) \circ m = \hbar \tilde{P})$  by its torsion ideal. Then  $\text{Mod}_{S_\hbar}(\underline{\text{Alg}}_{\text{qcom}})$  is the category of topologically free, cocommutative  $\mathbb{K}[[\hbar]]$ -algebras.

To describe  $\underline{\text{Alg}}_{\text{qcom}}$ , we use the following remark. Let  $M$  be a complete  $\mathbb{K}[[\hbar]]$ -module. Let us denote by  $M_\hbar$  its localization in  $\hbar$ ; this is a  $\mathbb{K}((\hbar))$ -vector space. Let  $M_{\text{tor}}$  be the torsion submodule of  $M$ , then  $M/M_{\text{tor}}$  is a  $\mathbb{K}[[\hbar]]$ -submodule of  $M_\hbar$ .

Let  $\underline{\text{Alg}}((\hbar))$  be the completed tensor product of  $\underline{\text{Alg}}$  with  $\mathbb{K}((\hbar))$ , i.e., the version "over  $\mathbb{K}((\hbar))$ " of  $\underline{\text{Alg}}$ . Then  $\underline{U} := \underline{\text{Alg}}((\hbar))\langle \tilde{P} \rangle / (m - (21) \circ m = \hbar \tilde{P})$  coincides with  $\underline{\text{Alg}}((\hbar))$ . On the other hand, the localization at  $\hbar$  of  $\underline{M} := \underline{\text{Alg}}[[\hbar]]\langle \tilde{P} \rangle / (m - (21) \circ m = \hbar \tilde{P})$  coincides with  $\underline{U}$ , therefore with  $\underline{\text{Alg}}((\hbar))$ . So for each  $(p, q)$ , the quotient  $\underline{\text{Alg}}_{\text{qcom}}(p, q) = \underline{M}(p, q) / \underline{M}(p, q)_{\text{tor}}$  is a  $\mathbb{K}[[\hbar]]$ -submodule of  $\underline{\text{Alg}}(p, q)((\hbar))$ .

**Proposition 6.1.** *Let  $\mathfrak{g}$  be the Lie algebra  $\mathcal{FL}_N^{\oplus n}$ . Denote by  $U(\mathfrak{g})^{\leq k}$  the linear span of the products of less than  $k$  elements of  $\mathfrak{g}$ , and by  $(U(\mathcal{FL}_N)^{\otimes n})^{\leq k}$  the image of  $U(\mathfrak{g})^{\leq n}$  under the identification  $U(\mathcal{FL}_N)^{\otimes n} = U(\mathfrak{g})$ . Then we have*

$$\underline{\text{Alg}}_{\text{qcom}}(N, n) \simeq \sum_{k \geq 0} \hbar^{k-N} (U(\mathcal{FL}_N)^{\otimes n})_{\sum_{i=1}^N \delta_i}^{\leq k} [[\hbar]]$$

as a submodule of  $\underline{\text{Alg}}(p, q)((\hbar))$ . The subscript still denotes the homogeneous component of degree  $\sum_{i=1}^N \delta_i$ .

*Proof.* Easy.  $\square$

By construction,  $\underline{\text{Alg}}_{\text{qcom}}(N, n)$  is a topologically free  $\mathbb{K}[[\hbar]]$ -module, and  $\underline{\text{Alg}}_{\text{qcom}}(N, n)/(\hbar)$  identifies with  $\oplus_{k \geq 0} \text{gr}_k(U(\mathcal{FL}_N)^{\otimes n})_{\sum_{i=1}^N \delta_i}$ , which by the PBW theorem identifies with

$$(S(\mathcal{FL}_N)^{\otimes n})_{\sum_{i=1}^N \delta_i},$$

i.e., with  $\underline{\text{Poisson}}(N, n)$ .

**Corollary 6.1.** *For any  $(N, n)$ , we have an isomorphism  $\underline{\text{Alg}}_{\text{qcom}}(N, n)/(\hbar) \xrightarrow{\sim} \underline{\text{Poisson}}(N, n)$ .*

We have a morphism  $\underline{\text{Poisson}} \rightarrow \underline{\text{Alg}}_{\text{qcom}}/(\hbar)$ , taking  $\eta, m$  to the reduction of their analogues and  $P$  to the reduction of  $\tilde{P}$  (this morphism is a counterpart of the functor taking the quasicommutative algebra  $A$  to the Poisson algebra  $A/\hbar A$ ).

Corollary 6.1 then shows that this is an isomorphism, so since  $\underline{\text{Alg}}_{\text{qcom}}$  is topologically free, we get

**Corollary 6.2.**  *$\underline{\text{Alg}}_{\text{qcom}}$  it is a flat deformation of  $\underline{\text{Poisson}}$ .*

*Remark 2.* Despite this fact, the props  $\underline{\text{Alg}}_{\text{qcom}}$  and  $\underline{\text{Poisson}}[[\hbar]]$  are not isomorphic. This can be checked explicitly. Besides, it is known that not any Poisson algebra can be quantized (see [Ma]).

**6.2. Props of formal series algebras.** We will say that a formal series commutative algebra is an augmented commutative algebra  $A$ , complete for the topology defined by the powers of its augmentation ideal  $\mathfrak{m}$ . A formal series Poisson algebra is such an algebra, equipped with a Poisson structure  $P$ , such that  $P(1, x) = 0$  and  $P(x, y) \in \mathfrak{m}$  for any  $x, y \in A$ . Finally, a formal series quasicommutative algebra is a quasicommutative augmented algebra over  $\mathbb{K}[[\hbar]]$ , topologically free as a  $\mathbb{K}[[\hbar]]$ -module, complete for the topology defined by the powers of  $\mathfrak{m}$ .

Define props of augmented commutative (resp., Poisson, quasicommutative) algebras as the props generated by  $\underline{\text{Alg}}_{\text{comm}}$  (resp.,  $\underline{\text{Poisson}}$ ,  $\underline{\text{Alg}}_{\text{qcom}}$ ), the generator  $\eta$  of bidegree  $(0, 1)$ , and the relations  $\epsilon \circ \eta = 0$ ,  $\eta \circ m = m \circ (\epsilon \otimes \epsilon)$ , together with: in the Poisson case  $\epsilon \circ P = 0$ , and in the quasicommutative case,  $\epsilon \circ \tilde{P} = 0$ . We denote them by  $\underline{\text{Aug}}_{\text{comm}}$ ,  $\underline{\text{Aug}}_{\text{Poisson}}$  and  $\underline{\text{Aug}}_{\text{qcom}}$ .

Then the corresponding props of formal series algebras are defined as the completions of these props with respect to the ideals  $I_k^{\text{comm}}$ ,  $I_k^{\text{Poisson}}$  and  $I_k^{\text{qcom}}$  generated by the  $m^{(l)} \circ (\text{id} - \eta \circ \epsilon)^{\otimes l}$ ,  $l \geq k$  in all three cases.

Before we describe these ideals, let us describe the props  $\underline{\text{Aug}}_X$ .

**Lemma 6.2.** *For any  $(N, n)$ , the canonical maps followed by composition with  $(\text{id} - \eta \circ \epsilon)^{\otimes N}$  induce isomorphisms*

$$\begin{aligned} \underline{\text{Alg}}_{\text{comm}}(N, n) &\simeq \underline{\text{Aug}}_{\text{comm}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N}, \quad \underline{\text{Poisson}}(N, n) \simeq \underline{\text{Aug}}_{\text{Poisson}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N}, \\ \underline{\text{Alg}}_{\text{qcom}}(N, n) &\simeq \underline{\text{Aug}}_{\text{qcom}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N}. \end{aligned}$$

We have therefore identifications

$$\begin{aligned} \underline{\text{Aug}}_{\text{comm}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N} &= (\mathcal{FC}_N^{\otimes n})_{\sum_i \delta_i}, \\ \underline{\text{Aug}}_{\text{Poisson}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N} &= (S(\mathcal{FL}_N)^{\otimes n})_{\sum_i \delta_i}, \\ \underline{\text{Aug}}_{\text{qcom}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N} &= \sum_{k \geq 0} \hbar^{k-N} (U(\mathcal{FL}_N)^{\otimes n})_{\sum_i \delta_i}^{\leq k}. \end{aligned}$$

We now describe the intersections of the ideals with these spaces. If  $\alpha \geq 1$ , we have

$$I_\alpha^{\text{comm}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N} = \oplus_{N \geq \alpha} (\mathcal{FC}_N^{\otimes n})_{\sum_i \delta_i},$$

$$I_\alpha^{\text{Poiss}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N} = \oplus_{k \geq \alpha} (S^k(\mathcal{FL}_N)^{\otimes n})_{\sum_i \delta_i},$$

$$I_\alpha^{\text{qcom}}(N, n) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N} = \hbar^{\alpha-N} (U(\mathcal{FL}_N)^{\otimes n})_{\sum_i \delta_i}[[\hbar]] \cap \sum_{k \geq 0} \hbar^{k-N} (U(\mathcal{FL}_N)^{\otimes n})_{\sum_i \delta_i}^{\leq k}[[\hbar]].$$

One checks that  $\underline{\text{Aug}}_{\text{qcom}}$  is a flat deformation of  $\underline{\text{Aug}}_{\text{Poiss}}$  and  $I_\alpha^{\text{qcom}}$  is a flat deformation of  $I_\alpha^{\text{Poiss}}$ , i.e., it is a saturated subspace whose reduction modulo  $\hbar$  coincides with  $I_\alpha^{\text{Poiss}}$ .

**6.3. Structures of  $\underline{\text{Bialg}}$  and of the related props.** Let  $X$  be one of the indices "no index", cP or coco. Then we have prop morphisms  $\underline{\text{Alg}} \rightarrow \underline{\text{Bialg}}_X$  and  $\underline{\text{Coalg}}_X \rightarrow \underline{\text{Bialg}}_X$ . Composition of these morphisms with the operation  $\circ$  of  $\underline{\text{Bialg}}_X$  induces linear maps  $\underline{\text{Coalg}}_X(p, N) \otimes \underline{\text{Alg}}(N, q) \rightarrow \underline{\text{Bialg}}_X(p, q)$ , which factor through the natural action of  $\mathfrak{S}_N$ .

**Proposition 6.2.** *The resulting linear maps*

$$i_{p,q} : \bigoplus_{N \geq 0} (\underline{\text{Coalg}}_X(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N} \rightarrow \underline{\text{Bialg}}_X(p, q)$$

are isomorphisms.

*Proof.* Let  $G$  be a graph for  $\underline{\text{Bialg}}_X(p, q)$ . Then the relations  $\Delta \circ m = (m \otimes m) \circ (1324) \circ (\Delta \otimes \Delta)$ , together with  $\delta \circ m = (m \otimes m) \circ (1324) \circ (\delta \otimes \Delta + \Delta \otimes \delta)$  when  $X = \text{cP}$ , imply that  $G$  can be transformed into a sum of graphs, where each operation  $\Delta$  (and  $\delta$  when  $X = \text{cP}$ ) occurs before each operation  $m$ . This proves that  $i_{p,q}$  is surjective.

Let us prove that  $i_{p,q}$  is injective. The structure of the prop  $\underline{\text{Alg}}$  implies that the map

$$i : \bigoplus_{N_1, \dots, N_q \geq 0} \underline{\text{Coalg}}_X(p, N_1 + \dots + N_q) \rightarrow \bigoplus_{N \geq 0} (\underline{\text{Coalg}}_X(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N},$$

taking  $\oplus_{N_1, \dots, N_q \geq 0} x_{N_1, \dots, N_q}$  to  $\oplus_{N \geq 0} y_N$ , where

$$y_N = \sum_{N_1, \dots, N_q \mid \sum_{i=1}^q N_i = N} x_{N_1, \dots, N_q} \otimes (m^{(N_1)} \otimes \dots \otimes m^{(N_q)}),$$

is a linear isomorphism. So we should prove that  $i_{p,q} \circ i$  is injective.

Let  $\mathcal{S} = \text{Vect}$ . We have a map  $\text{Mod}_{\mathcal{S}}(\underline{\text{Coalg}}_X) \rightarrow \text{Mod}_{\mathcal{S}}(\underline{\text{Bialg}}_X)$ , taking a  $X$ -coalgebra  $C$  to  $F(C)$ . Here  $F(C)$  is the free associative algebra over the vector space  $C$ , equipped with the unique algebra morphism  $\Delta_{F(C)} : F(C) \rightarrow F(C)^{\otimes 2}$  extending  $\Delta : C \rightarrow C^{\otimes 2}$ , and when  $X = \text{cP}$ , with the unique derivation  $\delta_{F(C)} : F(C) \rightarrow F(C)^{\otimes 2}$  extending  $\delta_C : C \rightarrow C^{\otimes 2}$ . Let  $x = \oplus_{N_1, \dots, N_q \geq 0} x_{N_1, \dots, N_q}$ , then  $(i_{p,q} \circ i)(x)_{F(C)}$  is a linear map  $F(C)^{\otimes p} \rightarrow F(C)^{\otimes q}$ . Composing it to the left with the  $p$ th power of the inclusion  $C \hookrightarrow F(C)$  and to the right with the tensor product of the projections  $F(C) \rightarrow C^{\otimes N_i}$ ,  $i = 1, \dots, q$ , we get a linear map  $C^{\otimes p} \rightarrow C^{\otimes N_1 + \dots + N_q}$ , which coincides with  $(x_{N_1, \dots, N_q})_C$ . This defines a linear map

$$\alpha_C : \underline{\text{Bialg}}_X(p, q) \rightarrow \bigoplus_{N_1, \dots, N_q \geq 0} \text{Hom}_{\mathcal{S}}(C^{\otimes p}, C^{\otimes N_1 + \dots + N_q}),$$

such that  $\alpha_C \circ (i_{p,q} \circ i)$  is the direct sum of the prop module maps  $\underline{\text{Coalg}}_X(p, N_1 + \dots + N_q) \rightarrow \text{Hom}_{\mathcal{S}}(C^{\otimes p}, C^{\otimes N_1 + \dots + N_q})$ . Taking  $C$  to be the cofree  $X$ -coalgebra with  $N_1 + \dots + N_q$  generators, we see that this map is injective. Therefore  $i_{p,q} \circ i$  is injective.  $\square$

**Corollary 6.3.** *Define the linear maps*

$$j_{p,q} : \bigoplus_{N \geq 0} (\underline{\text{Coalg}}_X(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N} \rightarrow \underline{\text{Bialg}}_X(p, q)$$

as the sum the maps taking  $x \otimes y$  to  $\iota_1(y) \circ (\text{id} - \eta \circ \epsilon)^{\otimes N} \circ \iota_2(x)$ , where  $\iota_1, \iota_2$  are the prop morphisms  $\underline{\text{Alg}} \rightarrow \underline{\text{Bialg}}_X$  and  $\underline{\text{Coalg}}_X \rightarrow \underline{\text{Bialg}}_X$ . Then  $j_{p,q}$  is a linear isomorphism.

*Proof.* Let us denote by  $\bigoplus_{N \geq 0} V_N$  the vector space on the left. One checks that  $j_{p,q} = i_{p,q} \circ k_{p,q}$ , where  $k_{p,q}$  is an endomorphism of  $\bigoplus_{N \geq 0} V_N$ , whose associated graded is the identity for the filtration defined by the  $\bigoplus_{N \leq k} V_N$ . So  $k_{p,q}$  is an isomorphism.  $\square$

The props  $\underline{\text{UE}}$  and  $\underline{\text{UE}}_{\text{cP}}$  are defined as completions of  $\underline{\text{Bialg}}_{\text{coco}}$  and  $\underline{\text{Bialg}}_{\text{cP}}$ . We therefore get:

**Proposition 6.3.** *The linear maps  $j_{p,q}$  extend to linear isomorphisms*

$$\widehat{j}_{p,q} : \widehat{\bigoplus_{N \geq 0} (\underline{\text{Coalg}}_{\text{coco}}(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N}} \rightarrow \underline{\text{UE}}(p, q)$$

and

$$\widehat{j}_{p,q} : \widehat{\bigoplus_{k \geq 0} \left( \bigoplus_{N \geq 0} (\underline{\text{Coalg}}_{\text{cP}}^k(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N} \right)} \rightarrow \underline{\text{UE}}_{\text{cP}}(p, q),$$

where  $\underline{\text{Coalg}}_{\text{cP}}^k(p, N) = S^k(\mathcal{FL}_N^{\oplus p})_{\sum_i \delta_i}$ .

The above arguments can be modified to show that the analogues of  $i_{p,q}$  and  $j_{p,q}$  define linear isomorphisms

$$\bigoplus_{N \geq 0} (\underline{\text{Coalg}}_{\text{qcoco}}(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N} \rightarrow \underline{\text{Bialg}}_{\text{qcoco}}(p, q).$$

We now obtain the structure of the prop  $\underline{\text{QUE}}$ . We define  $\underline{\text{Coalg}}_{\text{qcoco}}^{\geq \alpha}(p, N) \subset \underline{\text{Coalg}}_{\text{qcoco}}(p, N)$  as the intersection

$$\hbar^{\alpha-N} (U(\mathcal{FL}_N)^{\otimes p})_{\sum_i \delta_i} [[\hbar]] \cap \sum_{k \geq 0} \hbar^{k-N} (U(\mathcal{FL}_N)^{\otimes p})_{\sum_i \delta_i}^{\leq k} [[\hbar]].$$

This space identifies with its dual counterpart  $I_{\text{qcom}}^{\alpha}(N, p)$ , which is the set of all universally defined linear maps  $\mathfrak{m}^{\otimes N} \rightarrow A^{\otimes p}$  with image contained in  $(\mathfrak{m}^{(p)})^{\alpha}$ , where  $\mathfrak{m}$  is the augmentation ideal of a quasicommutative formal series algebra  $A$  and  $\mathfrak{m}^{(p)}$  is the augmenation ideal of  $A^{\otimes p}$ .

**Proposition 6.4.**  *$j_{p,q}$  extends to a linear isomorphism*

$$\widehat{j}_{p,q} : \lim_{\leftarrow \alpha} \left( \bigoplus_{N \geq 0} (\underline{\text{Coalg}}_{\text{qcoco}}(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N} / \bigoplus_{N \geq 0} (\underline{\text{Coalg}}_{\text{qcoco}}^{\geq \alpha}(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N} \right) \rightarrow \underline{\text{QUE}}(p, q).$$

*Proof.* Let  $\mathcal{I}_{\alpha}$  be the ideal of  $\underline{\text{Bialg}}_{\text{qcoco}}$  generated by the  $(\text{id} - \eta \circ \epsilon)^{\otimes \beta} \circ \Delta^{(\beta)}$ ,  $\beta \geq \alpha$ . Let  $\mathcal{I}_{\alpha}^{\text{sat}}$  be its saturation. We have  $\underline{\text{QUE}} = \lim_{\leftarrow \alpha} \underline{\text{Bialg}}_{\text{qcoco}} / \mathcal{I}_{\alpha}^{\text{sat}}$ .

We should prove that for each  $\alpha$ , the ideal  $\mathcal{I}_{\alpha}^{\text{sat}}$  is equal to the image  $\mathcal{J}_{\alpha}$  of  $\bigoplus_{N \geq 0} (\underline{\text{Coalg}}_{\text{qcoco}}^{\geq \alpha}(p, N) \otimes \underline{\text{Alg}}(N, q))_{\mathfrak{S}_N}$  in  $\underline{\text{Bialg}}_{\text{qcoco}}(p, q)$  under the map  $j_{p,q}$ . The inclusion  $\mathcal{I}_{\alpha}^{\text{sat}} \subset \mathcal{J}_{\alpha}$  is clear, so let us show the opposite inclusion.

Define  $\mathcal{I}'_{\alpha}$  as the ideal of  $\underline{\text{Bialg}}$  generated by all elements of the form  $(\text{id} - \eta \circ \epsilon)^{\otimes \beta} \circ \xi$ , where  $\xi \in \underline{\text{Coalg}}_{\text{qcom}}(1, \beta)$  and  $\beta \geq \alpha$ . Then  $\mathcal{I}'_{\alpha} \subset \mathcal{I}_{\alpha}^{\text{sat}}$ . We will prove that  $\mathcal{J}_{\alpha} \subset \mathcal{I}'_{\alpha}$ .

Set  $\delta^{(2)} = (\text{id} - \eta \circ \epsilon)^{\otimes 2} \circ \Delta$  and  $\tilde{\delta}^{(2)} = (\text{id} - \eta \circ \epsilon)^{\otimes 2} \circ \tilde{\delta}$ . Then the key relations are

$$\begin{aligned} \delta^{(2)} \circ m &= (m \otimes \text{id}) \circ (132) \circ (\delta^{(2)} \otimes \text{id}) + (\text{id} \otimes m) \circ (\delta^{(2)} \otimes \text{id}) + (m \otimes \text{id}) \circ (\text{id} \otimes \delta^{(2)}) \\ &\quad + (\text{id} \otimes m) \circ (213) \circ (\text{id} \otimes \delta^{(2)}) + (m \otimes m) \circ (1324) \circ (\delta^{(2)} \otimes \delta^{(2)}) \end{aligned}$$

and

$$\begin{aligned}\tilde{\delta}^{(2)} \circ m &= (m \otimes \text{id}) \circ (132) \circ (\tilde{\delta}^{(2)} \otimes \text{id}) + (\text{id} \otimes m) \circ (\tilde{\delta}^{(2)} \otimes \text{id}) + (m \otimes \text{id}) \circ (\text{id} \otimes \tilde{\delta}^{(2)}) \\ &+ (\text{id} \otimes m) \circ (213) \circ (\text{id} \otimes \tilde{\delta}^{(2)}) + (m \otimes m) \circ (1324) \circ (\tilde{\delta}^{(2)} \otimes \delta^{(2)}) \\ &+ (m \otimes m) \circ (1324) \circ ((21) \circ \delta^{(2)}) \otimes \tilde{\delta}^{(2)}.\end{aligned}$$

These relations allow one to show that for any  $x \in (\text{id} - \eta \circ \epsilon)^{\otimes \alpha} \circ \underline{\text{Coalg}}_{\text{qcoco}}(1, \alpha)$ ,  $x \circ m$  expressed as a sum  $\sum_{\beta, \gamma | \beta \geq \alpha \text{ or } \gamma \geq \alpha} X \otimes (Y \otimes Z)$ , where  $X \in \underline{\text{Alg}}(\beta + \gamma, 1)$ ,  $Y \in (\text{id} - \eta \circ \epsilon)^{\otimes \beta} \circ \underline{\text{Coalg}}_{\text{qcoco}}(1, \beta)$  and  $Z \in (\text{id} - \eta \circ \epsilon)^{\otimes \gamma} \circ \underline{\text{Coalg}}_{\text{qcoco}}(1, \gamma)$ . These relations allow one to arrange a diagram containing an element of  $(\text{id} - \eta \circ \epsilon)^{\otimes \alpha} \circ \underline{\text{Coalg}}_{\text{qcoco}}(1, \alpha)$  as a sum of *ordered* diagrams (i.e., of the form "algebra operations  $\circ$  coalgebra operations"), where all the coalgebra operations are in  $(\text{id} - \eta \circ \epsilon)^{\otimes \beta} \circ \underline{\text{Coalg}}_{\text{qcoco}}(1, \beta)$ ,  $\beta \geq \alpha$ .  $\square$

This result, the second part of Proposition 6.3, and Corollary 6.2 imply:

**Corollary 6.4.** QUE is a flat deformation of UE<sub>CP</sub>.

This is a consequence of Proposition 4.1, but the present proof of this fact does not use the existence of quantization functors.

*Remark 3.* Let us describe the spaces UE<sub>CP</sub>(1, 1) and QUE(1, 1). We will view them as spaces of maps  $\mathcal{O} \rightarrow \mathcal{O}$ , where  $\mathcal{O}$  is a Poisson formal series Hopf algebra in the first case and a quantized formal series Hopf algebra in the second. We define a "Poisson" array of operations

$$\begin{array}{ccc} x & x^{(1)}x^{(2)} & x^{(1)}x^{(2)}x^{(3)} & x^{(1)}x^{(2)}x^{(3)}x^{(4)} \\ \{x^{(1)}, x^{(2)}\} & \{x^{(1)}, x^{(2)}\}x^{(3)} \dots & \{x^{(1)}, x^{(2)}\}x^{(3)}x^{(4)} \dots \\ & \{\{x^{(1)}, x^{(2)}\}, x^{(3)}\} \dots & \{x^{(1)}, x^{(2)}\}\{x^{(3)}, x^{(4)}\} \dots \\ & & \{x^{(1)}, x^{(2)}\}\{x^{(3)}, x^{(4)}\} \dots \end{array}$$

and a "quantum" array

$$\begin{array}{ccc} f & f^{(1)}f^{(2)} & f^{(1)}f^{(2)}f^{(3)} & f^{(1)}f^{(2)}f^{(3)}f^{(4)} \\ [f^{(1)}, f^{(2)}]_{\hbar} & [f^{(1)}, f^{(2)}]_{\hbar}f^{(3)} \dots & [f^{(1)}, f^{(2)}]_{\hbar}f^{(3)}f^{(4)} \dots \\ & [[f^{(1)}, f^{(2)}]_{\hbar}, f^{(3)}]_{\hbar} \dots & \{f^{(1)}, f^{(2)}\}_{\hbar}[f^{(3)}, f^{(4)}]_{\hbar} \dots \\ & & [f^{(1)}, f^{(2)}]_{\hbar}[f^{(3)}, f^{(4)}]_{\hbar} \dots \end{array}$$

The dots indicate that other monomials belong to a given box, and  $[-, -]_{\hbar} = \frac{1}{\hbar}[-, -]$ . The bidegree  $(i, j)$  "Poisson box" consists of a basis of all polynomials of degree  $j$  in the  $x^{(\alpha)}$ , containing  $i - 1$  Poisson brackets. The Poisson array is graded by the diagonals parallel to the main diagonal. The quantum array is filtered by subspaces lying above the main diagonal (the spaces  $\underline{\text{Coalg}}_{\text{qcom}}^{\geq \alpha}(1, 1)$ ). An element of UE<sub>CP</sub>(1, 1) is an operation  $x \mapsto \sum_{k \geq 0} \sum$  (finite number of elements of the  $k$ th diagonal), where the first sum is infinite. An element of QUE(1, 1) is defined as a similar operation  $f \mapsto \sum_{k \geq 0} \sum$  (finite number of elements above the  $k$ th diagonal).  $\square$

**6.4. The prop of Lie bialgebras and propic Milnor-Moore theorems.** Define LBA as the prop with generators  $\mu, \delta$  with bidegrees  $(2, 1)$ ,  $(1, 2)$ , and relations

$$\mu + \mu \circ (21) = 0, \quad \mu \circ (\mu \otimes \text{id}) \circ ((123) + (231) + (312)) = 0, \quad (7)$$

$$\delta + (21) \circ \mu = 0, \quad ((123) + (231) + (312)) \circ (\delta \otimes \text{id}) \circ \delta = 0, \quad (8)$$

$$\delta \circ \mu = ((12) - (21)) \circ (\text{id} \otimes \mu) \circ (\delta \otimes \text{id}) \circ ((12) - (21)).$$

Then if  $\mathcal{S} = \text{Vect}, \text{Mod}_{\mathcal{S}}(\underline{\text{LBA}})$  is the category of Lie bialgebras over  $\mathbb{K}$ .



Define  $\underline{\mathbf{LA}}$  as the prop of Lie algebras, and  $\underline{\mathbf{LCA}}$  as the prop of Lie coalgebras. Then  $\underline{\mathbf{LA}}$  is generated by  $\mu$  of bidegree  $(2, 1)$  and relations (7),  $\underline{\mathbf{LCA}}$  is generated by  $\delta$  of bidegree  $(1, 2)$  and relations (8).  $\underline{\mathbf{LA}}$  (resp.,  $\underline{\mathbf{LCA}}$ ) corresponds to the operad (resp., cooperad) of Lie algebras (resp., coalgebras). We have

$$\underline{\mathbf{LA}}(N, n) = \underline{\mathbf{LCA}}(n, N) = (\mathcal{FL}_N^{\otimes n})_{\sum_{i=1}^N \delta_i}.$$

Moreover, in [Enr, Po], it is shown that the natural prop morphisms  $\underline{\mathbf{LA}} \rightarrow \underline{\mathbf{LBA}}$  and  $\underline{\mathbf{LCA}} \rightarrow \underline{\mathbf{LBA}}$  induce for each  $(p, q)$ , an isomorphism

$$\bigoplus_{N \geq 0} (\underline{\mathbf{LCA}}(p, N) \otimes \underline{\mathbf{LA}}(N, q))_{\mathfrak{S}_N} \xrightarrow{\sim} \underline{\mathbf{LBA}}(p, q).$$

Therefore, we have an isomorphism

$$\underline{\mathbf{LBA}}(p, q) \simeq \bigoplus_{N \geq 0} ((\mathcal{FL}_N^{\otimes p})_{\sum_{i=1}^N \delta_i} \otimes (\mathcal{FL}_N^{\otimes q})_{\sum_{i=1}^N \delta_i})_{\mathfrak{S}_N}. \quad (9)$$

On the other hand, we have:

**Theorem 6.1.** (*Propic Milnor-Moore theorem*) *We have a prop isomorphism  $\underline{\mathbf{UE}} \xrightarrow{\sim} S^*(\underline{\mathbf{LA}})$ , where  $S^* = \bigoplus_{i \geq 0} S^i$  is the "symmetric algebra" Schur functor.*

The proof of this theorem is based on the construction of "Eulerian idempotents".

**Lemma 6.3.** ([Lo]) *Let us define the rational numbers  $(\lambda_n^{(m)})_{n, m \geq 0}$  as the coefficients of the Taylor expansions at zero of  $\frac{1}{m!}(\ln(1+u))^m$ , so*

$$\frac{1}{m!}(\ln(1+u))^m = \sum_{n \geq 0} \lambda_n^{(m)} u^n.$$

For each  $m$ , the series

$$p_m = \sum_{n \geq 0} \lambda_n^{(m)} m^{(n)} \circ (\text{id} - \eta \circ \epsilon)^{\otimes n} \circ \Delta^{(n)}.$$

makes sense in  $\underline{\mathbf{UE}}(1, 1)$ , and the family  $p_m$  is a complete family of orthogonal idempotents, that is  $p_m p_{m'} = \delta_{m, m'} p_m$ , and the sum  $\sum_{m \geq 0} p_m$  is equal to  $\text{id}$ .

Moreover, if  $\mathfrak{g}$  is a Lie algebra, then  $U(\mathfrak{g})$  is a  $\underline{\mathbf{UE}}$ -module. Then  $(p_m)_{\mathfrak{g}} \in \text{End}(U(\mathfrak{g}))$  corresponds to the projection on the  $m$ th summand of  $\bigoplus_{i \geq 0} S^i(\mathfrak{g})$ , under the isomorphism  $\text{Sym}^{-1} : U(\mathfrak{g}) \rightarrow S^*(\mathfrak{g})$  (see [Lo]).

*Proof of Theorem 6.1.* If  $p, q, r$  are nonnegative integers, let  $\mathcal{FL}_{p+q}$  be the free Lie algebra with generators  $x_1, \dots, x_p, y_1, \dots, y_q$ . Then  $\text{Sym}(x_1 \otimes \dots \otimes x_p)$  and  $\text{Sym}(y_1 \otimes \dots \otimes y_q)$  belong to  $U(\mathcal{FL}_{p+q})$ . So does their product, and it is homogeneous of degree 1 in each generator.

Let  $m_{p,q}^r$  the image of this product in  $S^r(\mathcal{FL}_{p+q})$  under the composition  $U(\mathcal{FL}_{p+q}) \xrightarrow{\text{Sym}^{-1}} S^*(\mathcal{FL}_{p+q}) \rightarrow S^r(\mathcal{FL}_{p+q})$ . Then  $m_{p,q}^r$  lies in  $\underline{\mathbf{LA}}(S^p \otimes S^q, S^r)$ , and it vanishes unless  $r \leq p+q$ . Then  $m := \sum_{p,q} \sum_{r=0}^{p+q} m_{p,q}^r$  belongs to  $S^*(\underline{\mathbf{LA}})(1, 1)$ . We define  $\Delta \in S^*(\underline{\mathbf{LA}})(1, 2)$  by the rule that  $\Delta_{p,q}^r$  vanishes unless  $r = p+q$ , and then coincides the propic version of the coproduct for symmetric algebras. We define  $\epsilon \in S^*(\underline{\mathbf{LA}})(1, 0)$  by  $\epsilon_i = \delta_{i,0}$  and  $\eta \in S^*(\underline{\mathbf{LA}})(0, 1)$  by  $\eta^i = \delta_{i,0}$ . Then we have a prop morphism  $\underline{\mathbf{UE}} \rightarrow S^*(\underline{\mathbf{LA}})$ , taking  $m, \delta, \eta, \epsilon$  to their analogues.

We now construct a prop morphism  $S^*(\underline{\mathbf{LA}}) \rightarrow \underline{\mathbf{UE}}$ . Let  $p, q$  be integers  $\geq 0$ , and let  $x \in S^*(\underline{\mathbf{LA}})(p, q)$ . We set  $x = \bigoplus_{k_i \geq 0, l_i \geq 0} x_{k_1, \dots, k_p}^{l_1, \dots, l_q}$ , where  $x_{k_1, \dots, k_p}^{l_1, \dots, l_q} \in \underline{\mathbf{LA}}(\bigotimes_{i=1}^p S^{k_i}, \bigotimes_{j=1}^q S^{l_j})$ . We

define the map  $S^*(\underline{\mathbf{LA}}) \rightarrow \underline{\mathbf{UE}}(p, q)$  to take  $x$  to

$$\sum_{k_1, \dots, k_p \geq 0} \sum_{l_1, \dots, l_q \geq 0} ((m^{(l_1)} \circ \text{sym}^{(l_1)}) \otimes \dots \otimes (m^{(l_q)} \circ \text{sym}^{(l_q)})) \circ \varphi(x_{k_1, \dots, k_p}^{l_1, \dots, l_q}) \circ ((p_1^{\otimes k_1} \circ \delta^{(k_1)}) \otimes \dots \otimes (p_1^{\otimes k_p} \circ \delta^{(k_p)})).$$

Here we denote by  $\text{sym}_l$  the image of the total symmetrizer  $\frac{1}{l!} \sum_{\sigma \in \mathfrak{S}_l} \sigma \in \mathbb{Q}\mathfrak{S}_l$  in  $\underline{\mathbf{UE}}(l, l)$ . We denote by  $\varphi : \underline{\mathbf{LA}} \rightarrow \underline{\mathbf{UE}}$  the prop morphism taking  $\mu$  to  $m - m \circ (21)$ . We define  $\delta^{(p)} \in \underline{\mathbf{UE}}(1, p)$  as  $(\text{id} - \eta \circ \epsilon)^{\otimes p} \circ \Delta^{(p)}$ .

This formula corresponds to the following fact. Let  $\mathfrak{g}$  be a Lie algebra, and assume that  $x$  belongs to  $\underline{\mathbf{LA}}(\otimes_{i=1}^p S^{k_i}, \otimes_{j=1}^q S^{l_j})$ . Then  $x_{\mathfrak{g}} \in \text{Hom}(\otimes_{i=1}^p S^{k_i}(\mathfrak{g}), \otimes_{j=1}^q S^{l_j}(\mathfrak{g}))$ . Let  $\pi_k : U(\mathfrak{g}) \rightarrow S^k(\mathfrak{g})$  and  $i_l : S^l(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  be the projection and injection maps attached to the isomorphism  $U(\mathfrak{g}) \simeq S^*(\mathfrak{g})$ . Then

$$(\otimes_{j=1}^q i_{l_j}) \circ x_{\mathfrak{g}} \circ (\otimes_{i=1}^p \pi_{k_i}) \in \text{Hom}(U(\mathfrak{g})^{\otimes p}, U(\mathfrak{g})^{\otimes q}),$$

and it is given by the composition of maps:

$$\begin{aligned} U(\mathfrak{g})^{\otimes p} &\xrightarrow{\otimes_{i=1}^p (p_1^{\otimes k_i} \circ \delta^{(k_i)})_{\mathfrak{g}}} S^{k_1}(\mathfrak{g}) \otimes \dots \otimes S^{k_p}(\mathfrak{g}) \\ &\xrightarrow{x_{\mathfrak{g}}} S^{l_1}(\mathfrak{g}) \otimes \dots \otimes S^{l_q}(\mathfrak{g}) \xrightarrow{\otimes_{j=1}^q (m^{(l_j)} \circ \text{sym}_{l_j})_{\mathfrak{g}}} U(\mathfrak{g})^{\otimes q}. \end{aligned}$$

When writing this diagram, we understand that for any  $k \geq 0$ ,  $(p_1^{\otimes k} \circ \delta^{(k)})_{\mathfrak{g}}$  maps  $U(\mathfrak{g})$  to  $S^k(\mathfrak{g})$ . The reason why it corresponds to the above formula is that if  $y \in \underline{\mathbf{LA}}(p, q)$ , then  $\varphi(y) \in \underline{\mathbf{UE}}(p, q)$  is such that the restriction of  $\varphi(y)_{\mathfrak{g}}$  to  $\mathfrak{g}^{\otimes p}$  is a map  $\mathfrak{g}^{\otimes p} \rightarrow \mathfrak{g}^{\otimes q}$ , which coincides with  $y_{\mathfrak{g}}$ .

One then checks that this is a prop morphism, inverse to  $\underline{\mathbf{UE}} \rightarrow S^*(\underline{\mathbf{LA}})$ .  $\square$

In the same way, one proves the co-Poisson version of this result:

**Theorem 6.2.** *We have a prop isomorphism  $\underline{\mathbf{UE}}_{\text{cP}} \xrightarrow{\sim} S^*(\underline{\mathbf{LBA}})$ , such that the natural diagram involving the props  $\underline{\mathbf{UE}}$ ,  $\underline{\mathbf{UE}}_{\text{cP}}$ ,  $S^*(\underline{\mathbf{LA}})$  and  $S^*(\underline{\mathbf{LBA}})$  commutes.*

Taking into account (9), this induces an isomorphism

$$\underline{\mathbf{UE}}_{\text{cP}}(p, q) \simeq \widehat{\oplus}_{k \geq 0} \oplus_{N \geq 0} \left( (S^k(\mathcal{FL}_N^{\oplus p}))_{\sum_{i=1}^N \delta_i} \otimes (S^*(\mathcal{FL}_N)^{\otimes q})_{\sum_{i=1}^N \delta_i} \right)_{\mathfrak{S}_N},$$

which is the composition with the tensor product of  $q$  symmetrization maps, of the isomorphism

$$\underline{\mathbf{UE}}_{\text{cP}}(p, q) \simeq \widehat{\oplus}_{k \geq 0} \oplus_{N \geq 0} \left( (S^k(\mathcal{FL}_N^{\oplus p}))_{\sum_{i=1}^N \delta_i} \otimes (\mathcal{FA}_N^{\otimes q})_{\sum_{i=1}^N \delta_i} \right)_{\mathfrak{S}_N}$$

given by Proposition 6.3.

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